Virtual Arrays, Part 2: Virtual Arrays and Coarrays

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1 Objective

In a previous technical memorandum [1] the antenna concepts of the phase center of an antenna and the virtual element (VE) associated with a transmit-receive (T-R) channel were discussed. We also introduced the notions of the far field and the parallel ray approximation. In this memo, we will build on these ideas to introduce the concept of the virtual array (VA), and to try to understand the virtues and limitations of the VA idea. We will show that various physical antenna array configurations and data collection protocols can generate useful VAs. Finally, we will discuss the confusion over the concepts of coarrays and VAs, and show that the VA is *not* the convolution of the T and R array configurations, while the coarray is.[1]

2 The Physical Configuration

Consider a radar system having a transmit (T) array antenna and a receive (R) array antenna. In general, the two are physically separate, though in many cases of interest they will be the same array. We assume both arrays are *compact* and that targets of interest are in the *far field* of each; see [1] for our definition of these terms. We also assume that the T and R arrays are *colocated*; this is as opposed to "widely separated" antennas. We are not aware of an accepted formal definition of the limit on T-R spacing that defines "colocated", but here we assume that spacing to be much less than the distance to any scatterer $\mathbf{P}$ of interest. While we could be more general, there is little of practical use to us to be gained by doing so.

Figure 1 illustrates the general configuration of interest [3]. A transmit array contains N_T elements at ranges R_{Tn}, $n = 0,\ldots,N_T-1$ from the scatterer $\mathbf{P}$ and spans a total dimension of D_T. A receive array is similar, with N_R elements arranged over a total dimension D_R at ranges R_{Rn}, $n = 0,\ldots,N_R-1$ from $\mathbf{P}$. D_T and D_R are both much less than any of the $\{R_{Tn}\}$ and $\{R_{Rn}\}$ ("compact"), as is the separation between the T and R arrays ("colocated"). As a consequence of these assumptions, we can also assume that

- The parallel ray approximation is valid [1];
- The nominal angle from the transmit array to $\mathbf{P}$ is approximately the same as that from the receive array to $\mathbf{P}$;

[1] Maybe or maybe not surprisingly, the IEEE standard for antenna definitions [2] does not define either of the terms "coarray" or "virtual array".

- The nominal ranges from each array to **P** are also approximately the same, so that …
- The transmit and receive propagation losses are equal as well

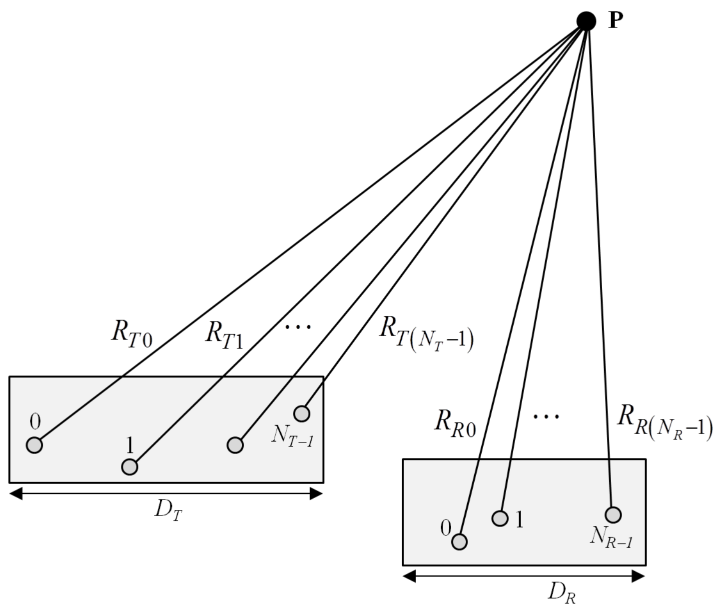

Figure 1. General compact, colocated transmit and receive arrays. The elements are not necessarily colinear or uniformly spaced, and the T and R arrays are not necessarily the same size.

Let the signal $x(t) = A \cdot m(t)\exp\left[j\left(2\pi F_0 t + \phi_0\right)\right]$ is emitted from transmit element n, where $m(t)$ is a baseband modulation function, for example a constant for a constant-frequency waveform, or a linear FM sweep or Barker phase code for a pulse compression waveform. The complex amplitude $\hat{A}_{nm}$ of the signal at receive element m can be found from

$$\begin{aligned} x''(t) &= \rho \cdot k^2 \cdot x\left(t - \left(R_{Tn} + R_{Rm}\right)/c\right) \\ &= \rho \cdot k \cdot A \cdot m\left(t - \left(R_{Tn} + R_{Rm}\right)/c\right) \cdot \exp\left[j\left(2\pi F_0\left(t - \left(R_{Tn} + R_{Rm}\right)/c\right) + \phi_0\right)\right] \\ &= \rho \cdot k^2 \cdot A \cdot m\left(t - \left(R_{Tn} + R_{Rm}\right)/c\right) \cdot \exp\left[j\left(2\pi F_0 t + \phi_0\right)\right]\exp\left[-j\left(2\pi\left(R_{Tn} + R_{Rm}\right)/\lambda\right)\right] \\ &= \rho \cdot \hat{A}_{nm} \exp\left[j\left(2\pi F_0 t + \phi_0\right)\right] \\ \Rightarrow \quad & \hat{A}_{nm} \equiv k^2 \cdot A \cdot m\left(t - \left(R_{Tn} + R_{Rm}\right)/c\right)\exp\left[-j\left(2\pi\left(R_{Tn} + R_{Rm}\right)/\lambda\right)\right] \end{aligned} \tag{1}$$

Here ρ is the reflectivity of the scatterer **P**. As in [1], k accounts for all amplitude factors due to one-way propagation, e.g. the $1/R$ amplitude loss and any atmospheric losses. Each T-R element pairing defines a single *channel* characterized in significant part by its two-way path length $R_{Tn} + R_{Rm}$ [1].

In this memo, we will be concerned primarily with one-dimensional linear arrays (LAs), often but not always shared for both transmit and receive, as shown in Figure 2. The left half of the figure shows a non-equispaced array of N elements. Let N be odd initially, $N = 2M + 1$, and index the elements from $-M$ to $+M$. Let the zero-indexed element be located at the origin on the x axis; note that this is not necessarily the center of the array extent D. The right half of the figure shows the common case of equispaced elements, called a *uniform linear array* (ULA). In this case the element locations are $x_n = nd$, the origin is at the center of the ULA, and the parallel ray approximation gives the range from each element to $\mathbf{P}$ as $R_n = R_0 - nd\sin\theta$, $n = -M, \cdots, +M$.

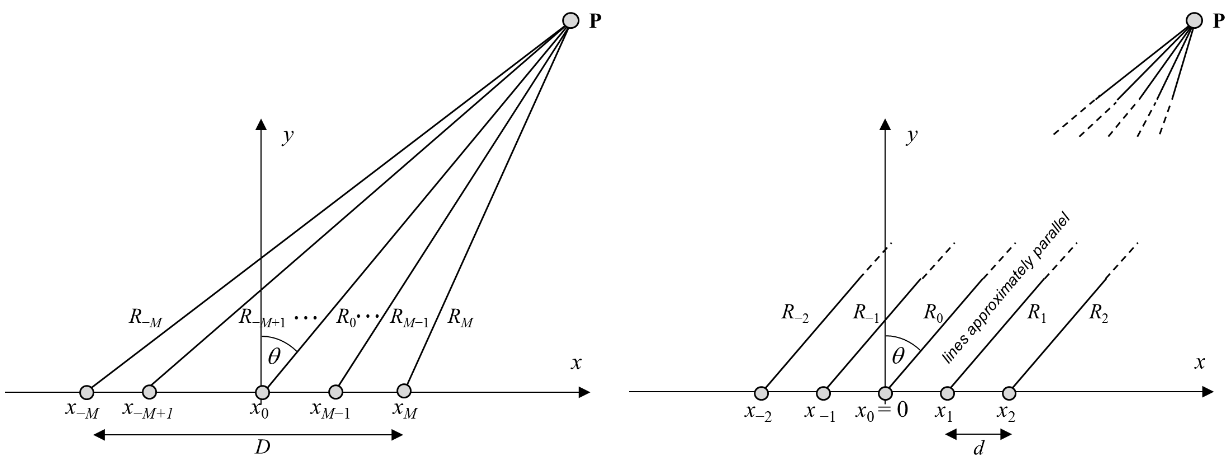

Figure 2. Compact one-dimensional linear arrays (LAs). In general, the elements are not uniformly spaced (left half of the figure). The uniform linear array (ULA) on the right has five equispaced elements indexed from -2 to $+2$ ($N = 5$, $M = 2$).

3 The Virtual Array

Given a transmit and receive array antenna, the corresponding *virtual array* (VA) is the assemblage of the *virtual elements* [1] corresponding to each T-R channel [4]. Even though there are only N_T+N_R elements, in general the VA will contain $N_R N_T$ VEs. Figure 3 illustrates the VA corresponding to the transmit and receive arrays of Figure 1. The left half of the figure shows the construction of the three VEs generated using the leftmost transmit element and the three receive elements; the VE is at the midpoint of the line connecting each T-R element pair. Also shown is the range from two of those VEs to $\mathbf{P}$. The right half shows the complete VA. The colors of the VEs that comprise the VA indicate the transmit element with which each is associated. There are $N_T N_R$ = 12 T-R channels, and so 12 VEs in the VA.

Note that the VA, as defined here, does *not* include any concept of directional antenna beams. Each original transmit and receive element, and each VE is assumed to radiate and receive isotropically.

Because it is built on the idea of VEs and phase centers, the VA is only intended to model phase data.

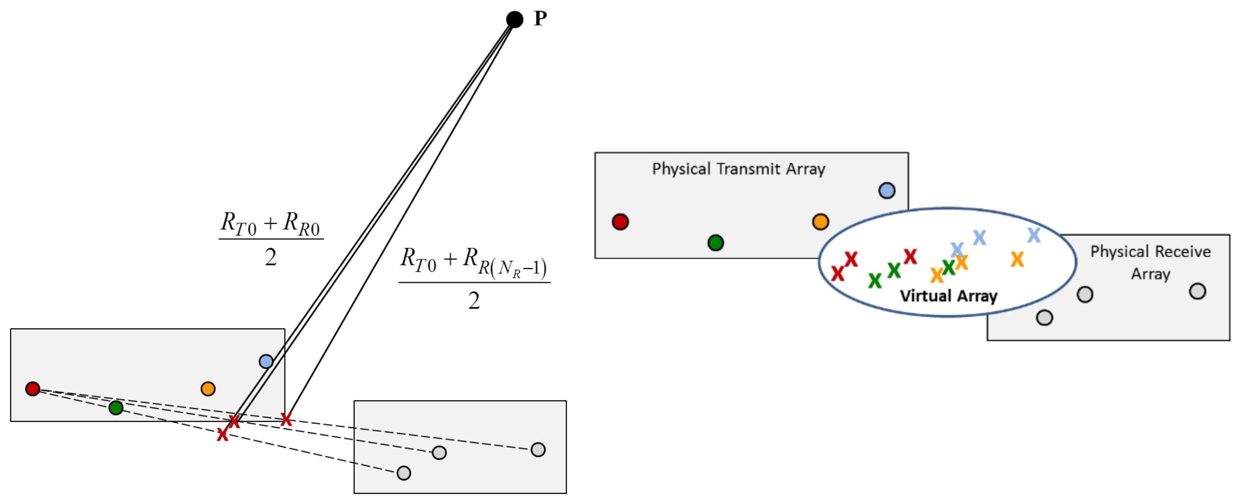

Figure 3. The virtual array. The left half of the figure illustrates construction of the location of the three virtual elements of the virtual array corresponding to the leftmost transmit element. The right half shows the complete virtual array. The colors indicate which VEs are associated with which transmit element.

It is easy to construct an array in which more than one channel results in the same VE location. This is the case, for instance, for any monostatic linear array having two or more element pairs, each located symmetrically about the same point. Each symmetric pair of elements then has a VE at that point. If N channels having a common VE location transmit different waveforms, there will simply be N coincident VEs using those different waveforms. If they transmit the same waveform, then they will effectively create a single VE transmitting that waveform with an amplitude that is N times higher than the transmission of a single physical element.

Given a physical T-R antenna configuration, the corresponding VA is unique. It is not entirely clear to me whether the converse is true or not, i.e. whether or not more than one T-R array pair can generate a given VA. In the degenerate case of only 1 transmit element and 1 receive element, the generating T-R pair is clearly not unique: any pair of T and R elements connected by a line having the same midpoint will generate the same VA, which is merely the single VE at that midpoint. Thus, rotating the T-R pair orientation around the midpoint, or increasing or decreasing the separation between them symmetrically about that midpoint, will leave the VA unchanged so long as the range to the target is large enough that the constraints of a compact array and therefore of far-field operation are satisfied. When the T and R arrays consist of multiple elements, however, it seems difficult to find a new set of T-R locations that generates the same set of VEs using the same number of elements.

4 Describing the Virtual Array: The Locator Function

Denote the set of N_R transmit element locations as $\{\mathbf{x}_{Tn}\}_{n=0,\ldots,N_T-1}$. It is convenient to represent this set of locations by a spatial "locator function" $L_T(\mathbf{x})$ that is simply a sum of Dirac impulse functions, one at each transmit element location: $L_T(\mathbf{x}) = \sum_{n=0}^{N_T-1} \delta(\mathbf{x}-\mathbf{x}_{Tn})$. Similarly, the locator function for the receive array is $L_R(\mathbf{x}) = \sum_{m=0}^{N_R-1} \delta(\mathbf{x}-\mathbf{x}_{Rm})$. The locator function for the virtual array is then simply the sum of impulse functions at each midpoint (VE location):

$$L_{TR}(\mathbf{x}) = \sum_{n=0}^{N_T-1} \sum_{m=0}^{N_R-1} \delta\left(\mathbf{x} - \frac{(\mathbf{x}_{Tn}+\mathbf{x}_{Rm})}{2}\right) \equiv L_{mid}(\mathbf{x}) \tag{2}$$

We will temporarily call this the *midpoint array*.

5 The Virtual Array as a Convolution, Not

Equation (2) is almost, but not quite, the convolution of the transmit and receive locator functions. To see this, write that convolution explicitly:

$$\begin{aligned} L_T(\mathbf{x}) * L_R(\mathbf{x}) &\equiv \int L_T(\mathbf{y}) L_R(\mathbf{x}-\mathbf{y})\,d\mathbf{y} \\ &= \int \left(\sum_{n=0}^{N_T-1} \delta(\mathbf{y}-\mathbf{x}_{Tn})\right)\left(\sum_{m=0}^{N_R-1} \delta(\mathbf{x}-\mathbf{y}-\mathbf{x}_{Rm})\right) d\mathbf{y} = \sum_{n=0}^{N_T-1}\sum_{m=0}^{N_R-1} \int \delta(\mathbf{y}-\mathbf{x}_{Tn})\,\delta(\mathbf{x}-\mathbf{y}-\mathbf{x}_{Rm})\,d\mathbf{y} \\ &= \sum_{n=0}^{N_T-1}\sum_{m=0}^{N_R-1} \delta(\mathbf{x}-(\mathbf{x}_{Tn}+\mathbf{x}_{Rm})) \\ &\equiv L_{sum}(\mathbf{x}) \end{aligned} \tag{3}$$

The last step used the identity $\int \delta(\mathbf{y}-\mathbf{y}_0) g(\mathbf{y})\,d\mathbf{y} = g(\mathbf{y}_0)$ [5]. The convolution produces a set of VE locations that is the sum of the transmit and receive element locations, without the scaling by 2 of Eq. (2). We will call this array, again temporarily, the *sum array*.

6 Are Coarrays and Virtual Arrays the Same Thing? And What is That Thing?

Some authors refer to a *coarray* (CA) associated with a transmit and receive array, in addition to or instead of a virtual array. Unfortunately, there appears to be some inconsistency in the literature regarding the definitions of the CA and VA, and whether or not they are the same thing. For example:

- Hoctor and Kassam [6] define the *sum coarray*[2] to be the set of element locations obtained as the sum of the transmit and receive element locations for all combinations of T and R elements

[2] They also define a *difference coarray*, but it is not applicable here.

(i.e., all T-R channels in our terminology). Consequently, their sum coarray is our sum array of Eq. (3).[3]

- Kilpatrick and Longstaff [7] define the term "coarray" to mean the sum array. They also discuss the midpoint array (called the "SAR array" in their discussion), pointing out the factor-of-two scaling between the two.
- Davis et al [8], Forsythe and Bliss [9], and Chen and P. P. Vaidyanathan [10] all also discuss the sum array, but assign it the name "virtual array".
- Finally, Davis [4] defines the term "virtual array" to mean the midpoint array.

Thus some authors define the virtual array as the sum array, while others define it as the midpoint array. Also, some authors use the term coarray, some virtual array, and some both.

Considering these examples as a whole, this author believes the best approach is to draw a distinction between the VA and CA. We will define the CA as the sum coarray obtained by convolution of the T and R arrays as in Eq. (3) and having locator function $L_{sum}(\mathbf{x})$. We will define the VA as the midpoint array having locator function $L_{mid}(\mathbf{x})$, as in Eq. (2). The coordinates of the CA VEs will then be those of the VA, scaled up by a factor of two: $L_{sum}(\mathbf{x}) = L_{mid}(2\mathbf{x})$. We will use the terms VA and CA (mostly VA) going forward, discarding the temporary terms "sum array" and "midpoint array". Finally, we will go back to referring to the locator function for the VA as $L_{TR}(\mathbf{x})$; thus, $L_{TR}(\mathbf{x}) = L_{mid}(\mathbf{x})$.

7 Interpreting the Virtual Array

How should we interpret a virtual array? Consider a set of transmit elements and a set of receive elements. In general, each transmit element transmits a different waveform at a different time, with each of those transmitted signals being scattered from a target scatterer $\mathbf{P}$ in the far field and received at each of the receive elements.[4] Each T-R pair thus describes a separate physical channel. The VA is imagined as an equivalent array consisting of the collection of corresponding independent monostatic virtual elements, one per physical channel. Two-way monostatic transmission by each VE duplicates the path length, and therefore received phase history, of one of the physical T-R channels. The ensemble of received signals observed by the complete VA will produce the same collective set of phase histories as that of the physical T-R configuration and data collection protocol.

In the case of the CA, the *one-way* path lengths from the CA VEs to $\mathbf{P}$ are the same as the two-way path lengths of the VA VEs. This suggests that, in order to produce the same set of received phase histories, the interaction of the coarray with $\mathbf{P}$ might be imagined as a *one-way* transmission from $\mathbf{P}$ located at $\mathbf{x} = \mathbf{0}$ to the full coarray acting as the receive array [5]. However, this viewpoint would seem to require a common transmit waveform, which we have labored *not* to assume so far.

[3] Hoctor and Kassam refer to the spatial configuration of the CA as the *morphological convolution* of the transmit and receive array spatial configurations [6].

[4] In practice, we will mostly be interested in less general scenarios.

An alternative interpretation of the CA, perhaps more palatable to a radar engineer's worldview, is to imagine a single transmit location that illuminates $\mathbf{P}$ with a single waveform. The reflected signal is received at the $N_T N_R$ elements of the CA. Each T-R path then includes a common term for the path length from the transmit source to $\mathbf{P}$. This term will produce an inconsequential common phase term in all of the received signals, which may be factored out into their complex amplitudes, leaving the *differences* in path lengths traveled as the essential information [4]. However, this viewpoint still seems to require a common transmit waveform.

8 Some Virtual Array Examples

The following subsections illustrate several VA examples. In all of these examples, we will restrict ourselves to uniform linear arrays for both transmit and receive, since most practical cases of interest to us fall into that class.

8.1 The Bistatic Case

Using the results above, we can describe the VA for the bistatic T-R configuration shown in the top line of Figure 4. The transmit "array" is a single element, $N_T = 1$. The receive array is a 1D ULA with and N_R = 5. The dashed lines show the transmit and receive element pairs that combine to form the left- and rightmost virtual elements of the VA on the third line, located at the midpoint between those pairs. The same process with the remaining receive VEs yields the complete VA shown. Both the VE spacing and the size of the VA are one-half those of the physical receive array. The VA is centered halfway between the T and R array centers. The number of distinct channels is N_R.

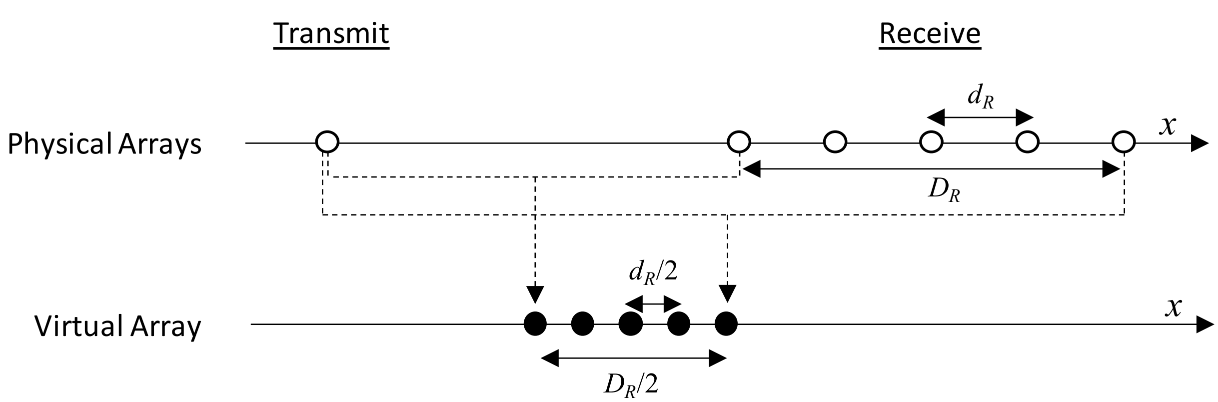

Figure 4. Physical and virtual arrays for a 1D bistatic ULA configuration.

8.2 Synthetic Aperture Radar

Synthetic aperture radar and sonar (SAR and SAS) is an imaging sensor technique for obtaining cross-range resolution much finer than can be obtained with a non-SAR or SAS "real beam" system. SAR concepts are discussed in [11]. Here, we are interested in the data collection protocol used by synthetic aperture sensors and the resulting virtual array.

A standard SAR system forms a synthetic array or aperture (SA) by physically locating a monostatic radar at one "element" location of the SA to be synthesized; radiating a pulse from that location; and collecting and storing the received data vs. range. The radar (presumed carried on a moving platform such as an aircraft or spacecraft in SAR, or towed by a submarine or surface ship in SAS) then advances to the next "element" location in the SA and radiates and receives another pulse. This continues indefinitely, with the effective SA size being determined by a "sliding window" style of processing that combines data from N consecutive transmit locations to define the SA extent.

Figure 5 constructs the VA for this mode of operation. Since each pulse transmission consists of transmitting and receiving from the same physical location,[5] each such transmission defines a T-R channel whose virtual element is also at the same location. That is, the VA is simply the assemblage of transmit locations that will be combined in subsequent processing. The number of distinct channels is again N.

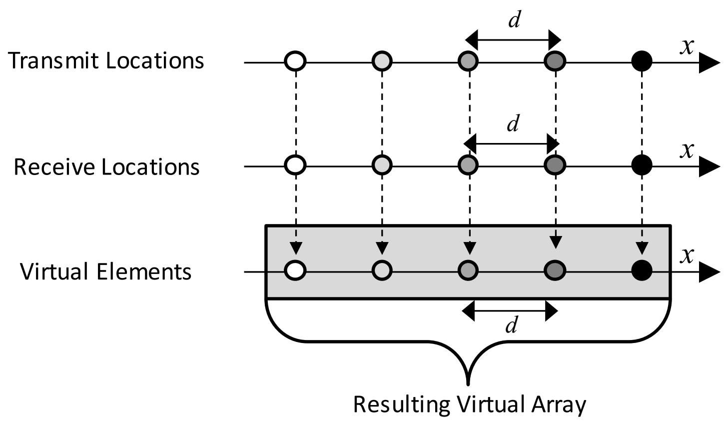

Figure 5. The virtual array in synthetic aperture operation. The shades of gray and dotted lines show which transmit and reception points define each virtual element in the VA.

It is well-known that a standard synthetic array of size D produces a beamwidth half as wide as that of the physical phased array of the same extent, thus obtaining 2× better angular resolution than the physical array [11]. This is easily explained by comparing the VAs for the two cases. Consider Figure 6. The left half constructs the VA of a system with a single transmit element and a co-located five-element receive ULA. This is essentially a monostatic version of Figure 4. The result is a virtual array one-half the extent of the physical array.

[5] The "stop and hop" approximation is usually invoked to justify ignoring motion of the platform during the pulse transit time. This is typically a very good assumption in SAR, less so in SAS. See [11] for a discussion.

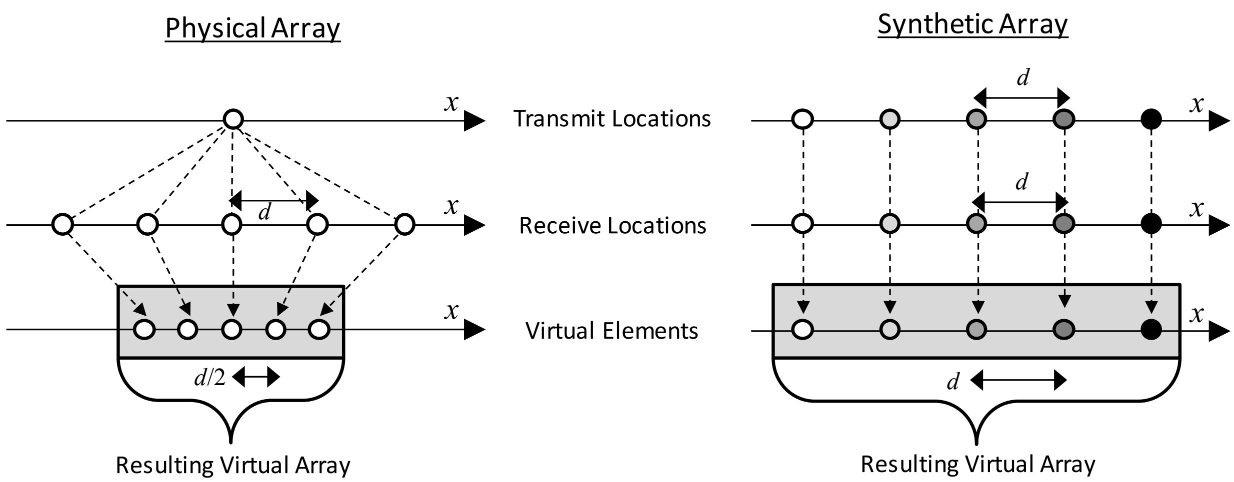

Figure 6. Virtual arrays for single-input physical and synthetic array formation. Dotted lines indicate which transmit and receive VE pairs form each VA element. Different shades of gray designate physical and virtual elements that correspond to the same pulse transmission: one pulse is transmitted in the physical array case, while five are transmitted in the synthetic array case.

The right half of the figure repeats the construction of Figure 5, resulting in a VA identical to the physical array. Therefore, the virtual array corresponding to a synthetic array of size D is twice the size of the VA corresponding to a physical array of the same size D, assuming a single transmit phase center. The larger effective antenna size of the SAR VA explains the reduced beamwidth and improved angular resolution of the synthetic array over the physical array. More differences between the two configurations will become apparent when we consider the antenna patterns in the next memo in this series.

8.3 The Vernier Array

Figure 7 shows an example of using the virtual array concept to provide properly equispaced samples in the along-track dimension for a synthetic aperture system. The upper half of the figure shows a 4-element physical array where the rightmost element can both transmit and receive. Transmission occurs only from that element; the other three elements are receive-only. The figure shows the position of the physical array at three consecutive pulse transmission times. The sensor platform and array move to the right ($+x$ direction) only; the displacement in the vertical direction is just for convenience in illustrating the overlapped array positions at those three times. The bottom half of the figure shows that the consecutive positions of the half-sized VA provide continuous, equispaced sampling in the x dimension.

This mode of operation is sometimes called a *vernier array*. It is common in synthetic aperture sonar, where the slow speed of propagation of sound in water may make it impossible to maintain a platform velocity slow enough to receive the transmission from one pulse before another is needed at the next desired spatial sampling location unless the range swath is kept very short. The vernier array allows more platform motion between transmissions while maintaining adequate spatial sampling.

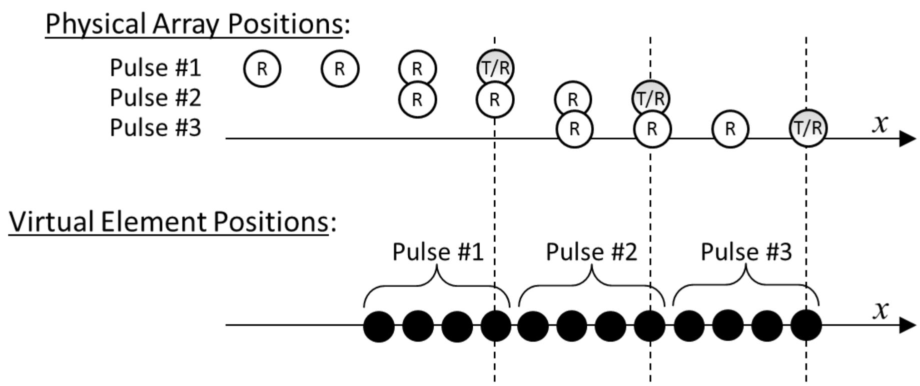

Figure 7. Operation of a physical SIMO arrays on consecutive pulses so as to provide continuous equispaced sampling. This technique is sometimes called a vernier array, and is common in synthetic aperture sonar.

8.4 Filling a Sparse Array

Figure 8 shows an example wherein a sparse physical array can result in a filled virtual array. The physical array has N_T = 3 transmitting elements which are operated separately, meaning either that they are operated sequentially or that they transmit different waveforms. It also has N_R = 3 receiving elements, one of which is common with one of the transmit elements. The colors of the virtual elements in the VA indicate the generating transmit element. The physical array is sparse in that it is has no transmit or receive elements at the two locations with dashed circles. The total number of physical elements is either N_T+N_R = 6 or $N_T+N_R-1 = 5$, depending on whether we count the shared T/R element as one or two elements. However, the resulting VA is *filled*, meaning that it has elements at uniform spacing with none missing. It is also non-redundant in that no two VEs exist at the same location. The total number of VEs in the VA is N_TN_R = 9. Thus, a filled VA can be created without the element cost of a completely filled physical array.

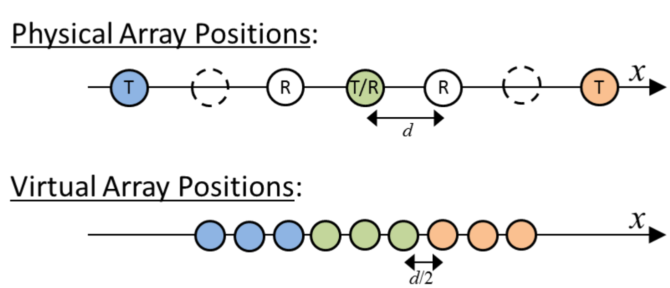

Figure 8. Creating a filled, non-redundant virtual array using a sparse physical array. The colors of the virtual elements in the VA indicate the generating transmit element.

8.5 Creating a Two-Dimensional Array from Two One-Dimensional Arrays

Two one-dimensional arrays, oriented at an angle to one another, create a two-dimensional virtual array. Figure 9 illustrates this for the case of 3-element transmit and receive ULAs (N_T,N_R = 3) oriented orthogonally to one another. The resulting VA is a filled 3×3 square array with $N_T N_R$ = 9 VEs, spaced by half the spacing of the T and R arrays.

Though not shown here, the element spacing in the T and R arrays does not have to be the same. If it isn't, the result would be a rectangular array. If the T and R arrays are not orthogonal in orientation, the resulting VA will be trapezoidal.

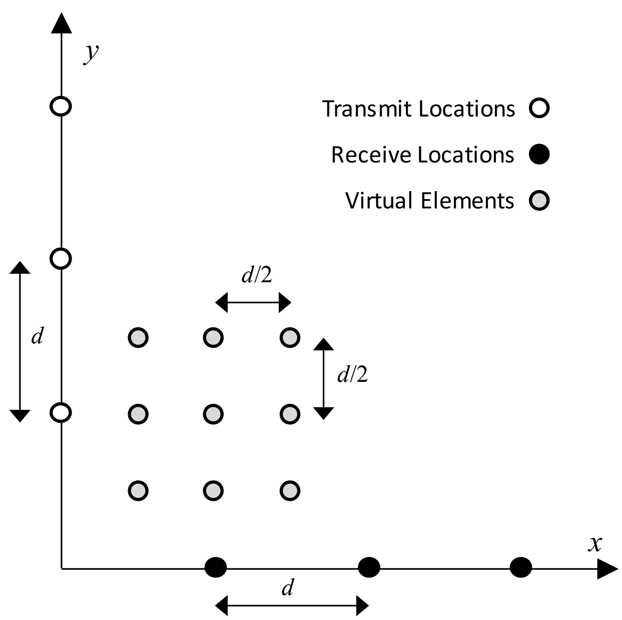

Figure 9. Creating a filled, non-redundant 2D virtual array from separate 1D transmit and receive arrays.

8.6 Consistency with Equation (2)

It is easily seen that formation of the VA in each of these example is consistent with Eq. (2). In the first bistatic example the transmit locator function is a single impulse $\delta(\mathbf{x}-\mathbf{x}_{T0})$. The VA locator function $L_{TR}(\mathbf{x})$ is then simply the receive locator function $L_R(\mathbf{x})$ shifted by $\mathbf{x}_{T0}$, with the resulting distribution then scaled down by a factor of two. The SAR case is a time series of sequential single-element T-R actions. Each action uses only a single "element" and therefore the T and R locator functions are identical, $L_T(\mathbf{x}) = L_R(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{x}_{T0})$. Each transmission then results in a VE at the same location. The VA is the assemblage of the time sequence of VEs and is therefore just the sum of the T-R locations, as illustrated in Figure 5.

The vernier array might be viewed as similar to a combination of the bistatic array and the SAR time series collection protocol. For the first transmission, the transmit locator function represents a single element at $L_T(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_{T0})$. The receive locator (using the 4-element example of Figure 7) is $L_R(\mathbf{x}) = \sum_{m=0}^{3} \delta(\mathbf{x} - \mathbf{x}_{T0} - md)$, where d is the receive element spacing. The VA for just the first transmission is then $L_{TR}(\mathbf{x}) = \sum_{m=0}^{3} \delta\left(\mathbf{x} - \left(\mathbf{x}_{T0} + (\mathbf{x}_{T0} - md)\right)/2\right) = \sum_{m=0}^{3} \delta\left(\mathbf{x} - \mathbf{x}_{T0} - m(d/2)\right)$, which is the half-size 4-element array labeled "Pulse #1" in the figure. On the second pulse, the physical array is shifted over by $N_R d/2 = 2d$ so that $L_T(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_{T0} - 2d)$, and the process is repeated. The complete VA is the assemblage of the component VAs created by each pulse transmission. The filled sparse array example calculations are much the same as those of the vernier array, with the difference that transmission occurs from all three transmit elements at the same time. The filled VA is then created on a single transmission.

For the two-dimensional example of Figure 9 we will write the locator function explicitly in terms of the two coordinates x and y. The transmit locator function is $L_T(x, y) = \sum_{m=1}^{3} \delta(0, y - md)$ and the receiver locator function is $L_R(x, y) = \sum_{n=1}^{3} \delta(x - nd, 0)$. Using Eq. (2), the resulting VA locator function is $L_{TR}(x, y) = \sum_{m=1}^{3} \sum_{n=1}^{3} \delta\left(x - m\frac{d}{2}, y - n\frac{d}{2}\right)$, which describes the result shown in the figure.

9 Virtual Arrays (huuuh!), What Are They Good For?

More than absolutely nothin'.

But why is the VA concept useful? For instance, does it simplify the calculation of important system characteristics, or lead to easier means of designing array systems to have certain properties? This is not clear to me yet. What we have seen is that:

- Given a physical transmit and receive array pair, the corresponding virtual array describes the effective set of spatial locations from which the target reflectivity is sampled. That is, independently operating a monostatic T-R element at each of the VA element locations will provide the same set of phase measurements as the physical configuration. This may be useful in analyzing spatial sampling requirements.
- Although we have only hinted at this through examples, it appears possible, at least in some cases, to design physical configurations that will provide desired sampling patterns while economizing on physical elements or other system characteristics.
- The virtual array for a given physical T-R array configuration is unique.
- The converse is not true; in at least some cases there can be more than one physical T-R array configuration that produces the same VA. I do not know under what conditions, if any, a particular VA can only be generated by a unique T-R physical array.

10 What's Missing?

The virtual array does not directly support the calculation of one- or two-way array factors; therefore it does not support calculation of antenna gains, mainlobe widths and angular resolution, or sidelobe structure, all important system characteristics. Calculating these metrics requires knowledge of the physical or virtual array geometry, but also knowledge of the manner in which the signals transmitted and received at each physical element (the "beamforming" utilized in the system) are combined. That will be the task of the next memo in this series [12].

11 References

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