Virtual Arrays and Coarrays, Part 1: Phase Centers and Virtual Elements

Mark A. Richards
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1 The Phase Center of an Antenna

The phase center of an antenna is defined as [1]

> *"The location of a point associated with an antenna such that, if it is taken as the center of a sphere whose radius extends into the far-field, the phase of a given field component over the surface of the radiation sphere is essentially constant, at least over that portion of the surface where the radiation is significant."*

This means that for a fixed range (the sphere radius), the phase should be independent of the angle(s) of the point of interest relative to the antenna orientation at any given time. This is equivalent to saying that the phase front (surface of constant phase) should be spherical. The phrase "*radius extends into the far-field*" means that the range at which the field phase is evaluated is in the far field of the antenna, generally taken as a range greater than $2d^2/\lambda$, where d is the antenna size and λ is the wavelength; see Section 6. The phrase "*at least over that portion of the surface where the radiation is significant*" means that the condition need be met only over the mainbeam of a conventional directive antenna pattern. The adjective "*essentially*" allows for an approximately, but not exactly, spherical phase front. The approximation tolerance is not specified in the definition.

We assume that we are interested in using coherent (quadrature, I/Q) receivers and signal processing, and so we model real sinusoidal signals using their complex equivalents throughout this memo [2]. A monochromatic electromagnetic (EM) wave is therefore modeled as having an electric field amplitude that varies with time at the source as $x(t) = A\exp\left[j\left(2\pi F_0 t + \phi_0\right)\right]$ for some frequency F_0 and initial phase ϕ_0. This wave propagates a distance R in R/c seconds. The observed E-field amplitude (the *signal*) at the end of the propagation path will be

$$\begin{aligned} x'(t) &= kA\exp\left[j\left(2\pi F_0\left(t - R/c\right) + \phi_0\right)\right] \\ &= kA\exp\left(-j2\pi F_0 R/c\right)\exp\left[j\left(2\pi F_0 t + \phi_0\right)\right] \\ &= kA\exp\left(-j2\pi R/\lambda\right)\exp\left[j\left(2\pi F_0 t + \phi_0\right)\right] \end{aligned} \tag{1}$$

where k accounts for all amplitude factors due to propagation, e.g. the $1/R$ amplitude loss and any atmospheric losses. Equation (1) shows that after propagating a distance R the signal will exhibit a phase shift of $-2\pi R/\lambda$ radians with respect to the original transmitted signal. This means that the echo phase will vary significantly with wavelength-scale changes in the range between the radar and a target or clutter patch, or between the radar and an interference source (e.g., a jammer). Many radar signal processing operations rely on modeling the specific spatial and temporal patterns of target echo, clutter echo, and interference phase shifts observed at the radar (the *phase histories*) to achieve their processing goals. Consequently, the pattern of phase shifts as a function of antenna geometry and echo source location and movement relative to the radar are of great interest.

2 Phase Center of an Isotropic Radiating Source

Consider an isotropically radiating source at coordinate $x = 0$ on the x axis as shown in Figure 1. The source emits an EM wave whose electric field amplitude is modeled as varying with time at the source as $x(t) = A\exp\left[j\left(2\pi F_0 t + \phi_0\right)\right]$. A point scatterer $\mathbf{P}$ is located a distance R_T away at an arbitrary angle θ with respect to the y coordinate as shown.

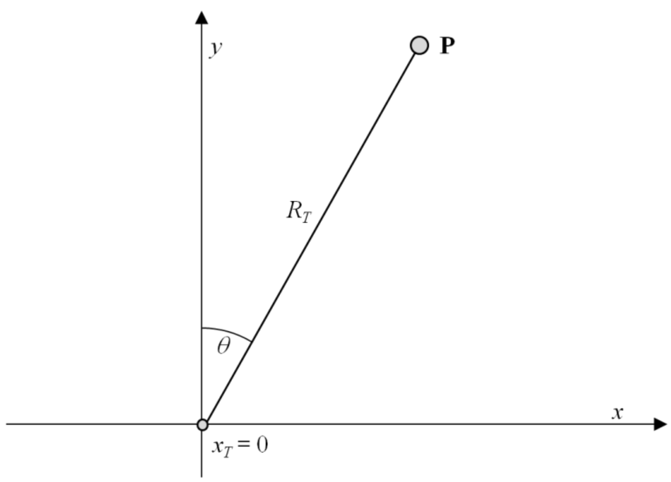

Figure 1. Geometry of a single radiating source and a distant scatterer.

The signal received at $\mathbf{P}$ will be

$$x'(t) = kA\exp\left(-j2\pi R_T/\lambda\right)\exp\left[j\left(2\pi F_0 t + \phi_0\right)\right] \tag{2}$$

The signal phase observed at $\mathbf{P}$ is simply the sum of the phases of the two exponential terms. This is independent of the angle θ, so by the definition above, the phase center of this "antenna" is simply the radiating source location.

A similar conclusion applies if we consider an echo reflected from $\mathbf{P}$ and received back at x_T. In this case the propagation distance is simply doubled, giving the received signal as

$$x''(t) = \alpha k^2 A \exp(-j4\pi R_T/\lambda)\exp\left[j(2\pi F_0 t + \phi_0)\right] \tag{3}$$

The constant α represents the effect on the amplitude of the scatterer itself, and so is related to the scatterer's radar cross section (RCS). We have implicitly assumed that α is independent of the aspect angle from which the incoming EM wave arrives, i.e. the scatterer is nonfluctuating with respect to aspect angle. The phase of the received signal is still independent of θ. Although the definition of phase center in the previous section contemplates only one-way propagation, we can generalize it a bit to include propagation from a transmitter, reflection from a scatterer, and reception at a receiver. For transmission from x_T and backscatter to a receiver at that same location, we can trivially say that the "phase center" for the transmit-receive (T-R) action is located at x_T.

In practice, the receiver will demodulate $x''(t)$ to obtain the received complex amplitude $\alpha k^2 A \exp\left[j(\phi_0 - 4\pi R_T/\lambda)\right]$. The initial transmit phase ϕ_0 obviously does not depend on θ, so to analyze the phase center of a given T-R element configuration, or even whether it has a defined phase center, it is sufficient in most of the remainder of this memorandum to consider just the T-R propagation path length and its dependence on θ. If that path length is some constant (or "essentially" constant) value $2R_{pc}$ over the desired range of θ (typically corresponding to the mainbeam of the configuration in question), then we can say that the phase center for that configuration is at a distance R_{pc} from $\mathbf{P}$.

3 A Transmit-Receive Element Pair and Their Phase Center

Now add a second, separate isotropic antenna element at coordinate x_R to serve as the receiver, forming a two-element linear array in the x coordinate. It is convenient to center the pair on $x = 0$, so if their separation is d, then $x_T = -d/2$ and $x_R = +d/2$ as shown in Figure 2. We define the angle θ with respect to a line of length R_C from the origin to $\mathbf{P}$. We assume $x_R > x_T$ (receiver to the right of the origin, transmitter to the left) and $0 \le \theta \le \pi/2$ for the moment, as shown. The T-R path length is $R_T + R_R$. Assume "closely spaced" transmit and receive elements in the sense that $d \ll R_C$, i.e. the distance to $\mathbf{P}$ is much greater than the separation of the two elements. (Generalizing, we will say an array is *compact* when its dimensions are small compared to the nominal distance to $\mathbf{P}$.) We also effectively assume that the scatterer's bistatic RCS is nonfluctuating with respect to the angles to both the transmit and receive elements.

We can compute the transmit and receive path lengths R_T and R_R using the law of cosines:

$$\begin{aligned} R_T^2 &= R_C^2 + (d/2)^2 - 2R_C(d/2)\cos(\theta + \pi/2) \\ &= R_C^2 + (d/2)^2 + R_C d\sin\theta \quad \Rightarrow \\ R_T &= R_C\sqrt{1 + \left(\frac{d}{2R_c}\right)^2 + \left(\frac{d}{R_c}\right)\sin\theta} \end{aligned} \tag{4}$$

Similarly,

$$R_R = R_C\sqrt{1 + \left(\frac{d}{2R_c}\right)^2 - \left(\frac{d}{R_c}\right)\sin\theta} \tag{5}$$

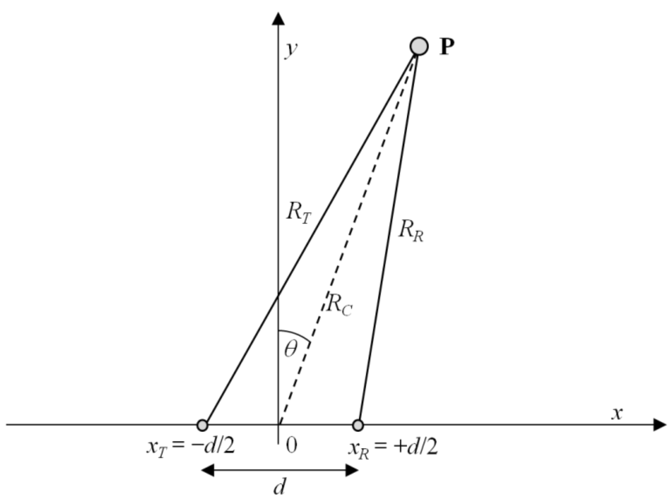

Figure 2. Geometry of two-element transmit-receive array.

The T-R path length $R_T + R_R$ determines the received phase. Define $\delta = (x_R - x_T)/R_C = d/R_C$ and write $R_T + R_R$ as follows:

$$\begin{aligned} R_T + R_R &= R_C\sqrt{1 + (\delta/2)^2 + \delta\sin\theta} + R_C\sqrt{1 + (\delta/2)^2 - \delta\sin\theta} \\ &= R_C\sqrt{1 + z_T} + R_C\sqrt{1 + z_R} \end{aligned} \tag{6}$$

where $z_T = (\delta/2)^2 + \delta\sin\theta$ and $z_R = (\delta/2)^2 - \delta\sin\theta$. The Maclaurin series for $\sqrt{1+z}$ is $1 + z/2 - z^2/8 + z^3/16 - \ldots$. Under the compact assumption, $|\delta| \ll 1$ and $|z_{T,R}| < \delta + \delta^2/4 \ll 1$ ($z_{T,R}$ refers to either z_T or z_R). We can therefore approximate the T-R path length by keeping only the first two terms of the series:

$$R_T + R_R \approx R_C\left\{2 + \frac{z_T + z_R}{2}\right\} = 2R_C\left\{1 + \frac{\delta^2}{8}\right\} \tag{7}$$

Repeating this analysis for the case where $x_R < x_T$ or for $-\pi/2 \le \theta \le 0$ shows that Eq. (7) applies for those cases also.

Equation (7) shows that the T-R path length is independent of θ (to within the accuracy of the approximation, which we will consider in the next section). Note that the equivalent sphere radius is larger than R_C, but only slightly so. It will be seen in Section 6 that in normal "far-field" operation, this difference is negligible.

For the record: including the $z^2/8$ term for greater accuracy would result in the second-order approximation

$$R_T + R_R \approx 2R_C\left\{1 + \left(\delta^2/8\right)\left(1 - \sin^2(\theta)\right) + \delta^4/128\right\} \approx 2R_C\left\{1 + \left(\delta^2/8\right)\left(1 - \sin^2(\theta)\right)\right\} \tag{8}$$

The $\sin^2\theta$ term introduces a slight dependence of the T-R path length on θ, showing that there is no exact, fixed phase center for the T-R pair, and that $x = 0$ is the phase center only up to the first order approximation; but again, that approximation is quite good in most practical circumstances.

4 The Virtual Element

Define the *virtual element* (VE) corresponding to a transmit-receive element pair to be a single element that emits the same waveform emitted by the transmit element, though possibly with a different amplitude; produces the same received complex amplitude as observed at a specified receive element or array; and is located at the phase center of the T-R element pair. For the single transmit and receive elements discussed so far, the VE exists; is located at the midpoint of the line connecting them ($x = 0$ in our example); and has the same transmit amplitude A as our transmit-only element. Also, notice that the roles of the transmit and receive elements can be reversed without changing the received amplitude or phase, or the location of the VE.

5 Channels

Borrowing some terminology from communications, we can call the path from a transmit source to a scatterer and then to a receive source a single *channel*. The results so far show that a bistatic channel such as that of Figure 2 is equivalent to a monostatic channel originating and ending at the VE. That is, both produce the same received waveform, specifically including the same phase shift of the received echo and, assuming still that the bistatic RCS is isotropic, the same amplitude. Again, the roles of the transmit and receive elements can be reversed without changing the received amplitude or phase.

6 Approximation Error and the Far-Field Condition

The error in approximating $R_T + R_R$ of Eq. (6) by the first-order approximation of Eq. (7) is R_C times the error in approximating the quantity $\left(\sqrt{1+z_T}+\sqrt{1+z_R}\right)$. Consider the error $\varepsilon(z)$ in replacing $\sqrt{1+z}$ by the first-order approximation $1+z/2$. Since this is the error in each of the two square root approximations, the upper bound on the error in approximating $R_T + R_R$ will be $R_C \cdot 2\max_z\left\{\left|\varepsilon(z)\right|\right\}$. The Lagrange form of the Taylor series remainder estimation theorem [3] gives us

$$\left|\varepsilon(z)\right| \le \left|\frac{d^2}{dz^2}\left(\sqrt{1+z}\right)\right| \cdot \frac{|z|^2}{2!} = \frac{|z|^2}{8|1+z|^{3/2}} \tag{9}$$

This error is maximum when $|z|$ takes on its maximum value. Since $z_{T,R} = (\delta/2)^2 \pm \delta\sin\theta$ and $\delta \ll 1$, the maximum value of either $|z_T|$ or $|z_R|$ is approximately δ (occurring when $\theta = \pm\pi/2$), giving

$$\left|\varepsilon(z)\right| \le \frac{\delta^2}{2(1+2\delta)^{3/2}} \le \frac{\delta^2}{2} \tag{10}$$

Require that the error in the estimate of $R_R + R_T$ be less than $\lambda/8$ so that the error in the two-way T-R phase is no more than 45°. Our condition becomes

$$2R_C\frac{\delta^2}{8} < \frac{\lambda}{8} \quad\Rightarrow\quad \delta < \sqrt{\frac{\lambda}{2R_C}} \quad\Rightarrow\quad d < \sqrt{\frac{\lambda R_C}{2}} \ \text{ or } \ R_C > \frac{2d^2}{\lambda} \tag{11}$$

Equation (11) is a typical expression of the antenna size limit for which the *Fraunhofer approximation* to the scalar diffraction expression of antenna radiation is valid [4]. The last form given in the equation is a common definition of the *far field distance* from an antenna [4]. In practical radar applications of interest, the range from each antenna to a scatterer will normally be far beyond this far-field distance, so going forward we will assume this is always true. This means that in replacing a T-R element pair by the corresponding virtual element, we are guaranteed that the errors in computing the phase of the received signal will be less (usually much less) than 45°.

7 Three-Dimensional Extension

Although developed in two dimensions, the conclusions so far are valid in three-dimensional space. No matter what the location of the transmit and receive elements and the scatterer **P** in 3D space, the coordinates can always be rotated and translated to place the transmit and receive elements on the x axis, symmetric around $x = 0$, and then to place **P** in the x-y plane as shown in Figure 2. The VE will then be at the midpoint of the line connecting the transmit and receive elements ($x = 0$ in the transformed coordinates).

8 The Parallel Ray Approximation

Consider the one-way path from x_T to $\mathbf{P}$ in comparison to the path length from the VE to $\mathbf{P}$. We know that $R_T = R_C\sqrt{1+z_T}$, with $z_T = (\delta/2)^2 + \delta\sin\theta$. The first-order Maclaurin series approximation to R_T is

$$R_T \approx R_C\left\{1+\frac{z_T}{2}\right\} = R_C\left\{1+\frac{\delta}{2}\sin\theta+\frac{\delta^2}{8}\right\} \tag{12}$$

We can simplify further if $\delta^2/8 \ll (\delta/2)\sin\theta \quad \Rightarrow \quad \delta \ll 4\sin\theta$. Then

$$R_T \approx R_C\left(1+\frac{\delta}{2}\sin\theta\right) = R_c + \frac{d}{2}\sin\theta \tag{13}$$

Similarly, $R_R \approx R_c - (d/2)\sin\theta$. The validity of this approximation will be considered shortly. Figure 3 shows that Eq. (13) is equivalent to assuming that $\mathbf{P}$ is far enough away that the vectors from x_T and the origin (the VE) are approximately parallel.

The two-way T-R path length under this "parallel ray" approximation is

$$R_T + R_R \approx 2R_C \tag{14}$$

which is simply the two-way path length from the VE to $\mathbf{P}$ and back. This approximation differs by $R_C\delta^2/4 = d^2/4R_C$ from the result of Eq. (7). In the usual far-field operation, this difference is less (usually much less) than $\lambda/8$ and is insignificant.

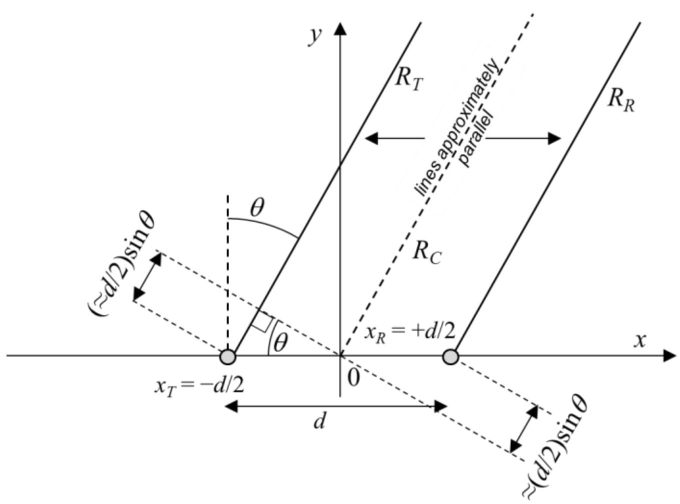

Figure 3. Parallel-line approximation to range difference for distant scatterers.

The parallel ray approximation assumption that $\delta \ll 4\sin\theta$ will fail for θ sufficiently near zero. In the very small region over which this typically occurs, we have instead from Eq. (12) $R_T \approx R_C\left\{1+\delta^2/8\right\}$.[1] Once more, in the far-field that difference between this result and that of Eq. (14) is not significant.

9 Sequels: Virtual Arrays, Coarrays, and MIMO Radar

In parts 2 and 3 of this memo series we will start with these results and use them to describe the ideas of virtual arrays and coarrays. Once we have figured those out, a final future memo in this series will build upon those ideas to introduce multiple-input, multiple-output (MIMO) radar arrays.

10 References

[1] "IEEE Standard Definitions of Terms for Antennas", IEEE Standard 145-1993 (R2004), 2013.
[2] M. A. Richards, *Fundamentals of Radar Signal Processing*, second edition. McGraw-Hill, 2014.
[3] "Taylor's theorem", https://en.wikipedia.org/wiki/Taylor's_theorem.
[4] R. E. Blahut, *Theory of Remote image Formation*. Cambridge University Press, 2004.

[1] Note that this is just the one-way equivalent of Eq. (7).