Swerling 2 = Swerling 3 When N = 2

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1 Background

In the course of exploring the ins and outs of noncoherent integration, the author recently observed, rather by accident, that the receiver operating characteristic (ROC) curves for the Swerling cases 2 and 3 having the same average single-sample signal-to-noise ratio (SNR) seemed to be the same when the number of samples noncoherently integrated was N = 2. This appears to be true for any value of false alarm probability P_{FA}, but only for N = 2. The following explanation of why this is the case draws on the results in Appendix A of [1].

2 Chi-Square Random Variables

The Swerling 2 model assumes an exponential probability density function (PDF) for each target sample. The Swerling 3 assumes a 4th-degree chi-square. Both are special cases of the general chi-square distribution[1]

$$p_x(x) = \chi^2\left(x; M, \sigma^2\right) = \begin{cases} \dfrac{x^{M/2-1}}{\left(2\sigma^2\right)^{M/2}\Gamma\left(\dfrac{M}{2}\right)} \exp\left(-x/2\sigma^2\right), & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{1}$$

having mean and variance

$$\bar{x} = M\sigma^2, \quad \sigma_x^2 = 2M\sigma^4 \tag{2}$$

where M is the number of degrees of freedom (also called the "duo-degree"). M = 2 for the exponential and 4 for the 4th-degree chi-square.

We are interested in integration (summing) of random variables, so the characteristic function (CF) is of interest.[2] The CF of the general chi-square is (substituting also $\sigma^2 = \bar{x}/M$ from (2))

[1] The general chi-square of Eq. (1) is the PDF of the sum of the squares of M independent, identically-distributed (iid) Gaussian random variables of zero mean and variance σ^2.
[2] The CF of the sum of N independent random variables is the product of their individual CFs. If the random variables are iid, this is just the CF of the single random variable raised to the N^{th} power.

$$C_{\chi^2}(q) = \frac{1}{\left(1 - j2\frac{\bar{x}}{M}q\right)^{M/2}} \tag{3}$$

3 Noncoherent Integration of Two Swerling 2 Variates

The Swerling 2 model assumes noncoherent integration of N iid exponential random variables. The relevant CF for noncoherent integration of N = 2 Swerling 2 variates is therefore the square of Eq. (3) with M = 2:

$$C_{\text{SW2}}(q) = \frac{1}{\left[\left(1 - j2\frac{\bar{x}}{M}q\right)^{M/2}\right]^2} = \frac{1}{\left(1 - j\bar{x}q\right)^2} \tag{4}$$

4 Noncoherent Integration of Two Swerling 3 Variates

The Swerling 3 model assumes a 4th-degree chi-square PDF, so M = 4. However, in the Swerling 3 decorrelation model the integrated samples are *not* independent draws from the PDF; they are simply N copies of the same random variable. The result is then N times the value of a single 4th-degree chi-square variates. For N = 2, this is a 4th-degree chi-square random variable with a mean value $2\bar{x}$. Therefore the relevant CF is Eq. (3) with M = 4, no squaring, and $\bar{x}$ doubled:

$$C_{\text{SW3}}(q) = \frac{1}{\left(1 - j2\frac{(2\bar{x})}{4}q\right)^{4/2}} = \frac{1}{\left(1 - j\bar{x}q\right)^2} = C_{\text{SW2}}(q) \tag{5}$$

Clearly, the CF and therefore the PDF is the same in both cases. Consequently, the ROC will also be the same, as originally observed.

5 References

[1] M. A. Richards, *Fundamentals of Radar Signal Processing*, second edition. McGraw-Hill, 2014.