Spatial and Temporal Frequency

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1 One-way Spatial Frequency

Figure 1 shows a receiver located at coordinates (x_r,y_r) and traveling in the +x dimension. The x direction is called the *along-track* dimension because it is in the same direction as the path ("track") followed by the receiver. The orthogonal dimension y is called the *cross-track* dimension.

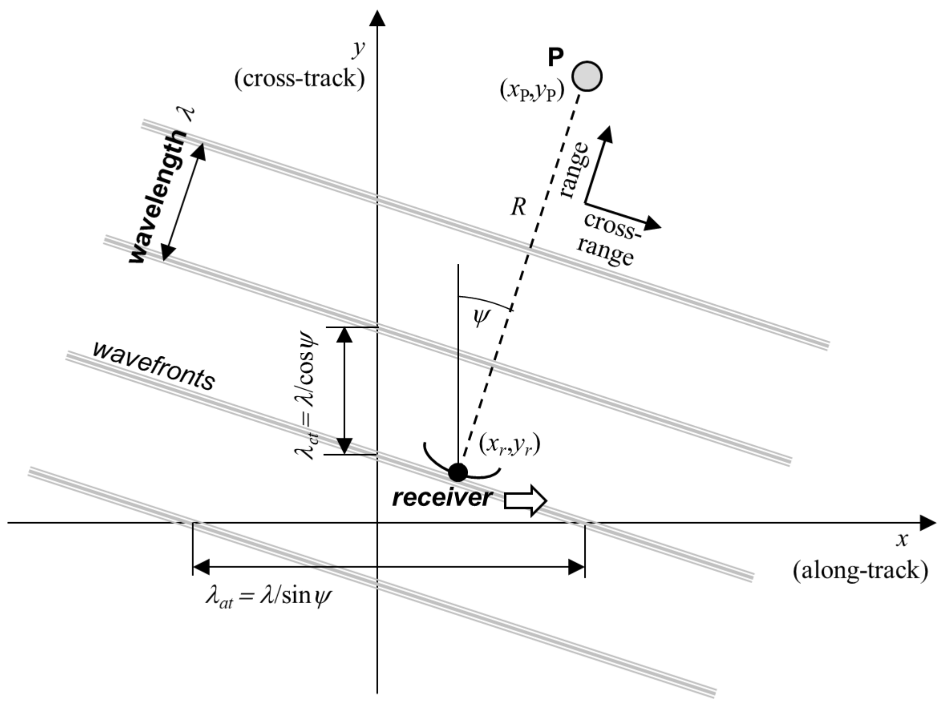

Figure 1. Geometry for computing spatial wavelengths and frequencies. See text for details.

Consider a monochromatic electromagnetic wave propagating from an emitter **P** at (x_P,y_P). **P** is at an angle ψ with respect to the receiver's cross-track normal as shown in Figure 1. The emitted signal is modeled as the general complex sinusoidal form $x(t) = A\exp\left[j(\Omega t + \phi)\right]$. Assume that **P** and the radar are far enough apart that the far-field approximation [1] applies and that the EM wave can be considered planar at the radar. The signal at the receiver is

$$\begin{aligned} y(t) &= A\exp\left[j\left(\Omega(t - R/c) + \phi\right)\right] \\ &= A\exp\left[j\left(\Omega t - (2\pi/\lambda)R + \phi\right)\right] \\ &\equiv A\exp\left[j\Phi(t,R)\right] \end{aligned} \tag{1}$$

where $R = \sqrt{\left(x_{\mathrm{P}} - x_r\right)^2 + \left(y_{\mathrm{P}} - y_r\right)^2}$ is the receiver-emitter range, c is the speed of light, λ is the signal wavelength, $\Phi(t,R)$ is called the *phase* of the received signal, and we have used $\Omega = 2\pi F = 2\pi c/\lambda$.

The EM plane wave oscillates periodically in both time and space, and so can be characterized by its temporal and spatial periods or, equivalently, frequencies. Temporal frequency is a single scalar value. Spatial frequency, however, is a directional quantity; its value depends on the direction of propagation and the spatial dimension along which it is evaluated.

The instantaneous frequency of a signal is defined as the derivative of its phase [2]. For example, computing the temporal frequency as the derivative of $\Phi(t,R)$ at a fixed range R with respect to t confirms that the instantaneous temporal frequency is Ω radians per second:

$$\Omega_i \equiv \frac{d\Phi}{dt} = \frac{d}{dt}\left\{\Omega t - \left(2\pi/\lambda\right)R + \phi\right\} = \Omega \text{ rads/sec } = 2\pi F \text{ Hz} \tag{2}$$

Now consider the instantaneous spatial frequency along the range dimension, that is, the line of sight (LOS) connecting the receiver and **P**. The instantaneous *one-way range spatial frequency* K is the derivative of $\Phi(t,R)$ at a fixed time t with respect to R:

$$K \equiv \frac{d\Phi}{dR} = \frac{d}{dR}\left\{\Omega t - \left(2\pi/\lambda\right)R + \phi\right\} = -\frac{2\pi}{\lambda} \text{ rads/m } = -\frac{1}{\lambda} \text{ cycles/m} \tag{3}$$

Notice that the units of spatial frequency are radians or cycles per meter, not per second. Equation (3) states that, at a given time, an observer moving along the LOS direction would observe one full cycle (2π radians or 360°) of phase shift for every λ meters of motion, consistent with a wavelength of λ meters along the range direction. It is customary to call the one-way spatial frequency in radians/meter the *wavenumber* and to denote it K.[1] Here we relabel the result of Eqn. (3) as K_R to denote wavenumber in the range dimension. We also denote spatial frequency in cycles/meter as F_R. Thus

$$K_R \equiv \frac{d\Phi}{dR} = -\frac{2\pi}{\lambda} \text{ rads/m,} \quad F_R \equiv \frac{1}{2\pi}\frac{d\Phi}{dR} = -\frac{1}{\lambda} \text{ cycles/m} \tag{4}$$

We can also compute the spatial frequency in the cross-track and along-track dimensions by expressing the range in terms of those dimensions. The instantaneous along-track spatial frequency is

[1] It is also customary to use a lower case k. I prefer to distinguish continuous ("analog") frequencies from discrete (sampled, "digital") frequencies by using upper case symbols for the former and lower case for the latter, in keeping with much of the digital signal processing literature [2],[3].

$$
\begin{aligned}
K_{at} &\equiv \frac{d\Phi}{dx_r} = \frac{d}{dx_r}\left\{\Omega t - (2\pi/\lambda)\sqrt{(x_P - x_r)^2 + (y_P - y_r)^2} + \phi\right\} \\
&= -\frac{2\pi}{\lambda}\cdot\frac{1}{2}\left\{(x_P - x_r)^2 + (y_P - y_r)^2\right\}^{-1/2}\cdot 2(x_P - x_r)\cdot(-1) \\
&= \frac{2\pi}{\lambda}\cdot\frac{(x_P - x_r)}{\sqrt{(x_P - x_r)^2 + (y_P - y_r)^2}} = \frac{2\pi}{\lambda}\cdot\frac{(x_P - x_r)}{R} \\
&= \frac{2\pi}{\lambda}\sin\psi \text{ rads/m;} \\
F_{at} &= \frac{1}{\lambda}\sin\psi \text{ cycles/m}
\end{aligned}
\tag{5}
$$

The one-way cross-track spatial frequency is obtained similarly as $d\Phi/dy_r$ and is

$$
K_{ct} \equiv \frac{d\Phi}{dy_r} = \frac{2\pi}{\lambda}\cos\psi \text{ rads/m;} \quad F_{ct} \equiv \frac{1}{\lambda}\cos\psi \text{ cycles/m} \tag{6}
$$

It will always be the case that $\sqrt{K_{at}^2 + K_{ct}^2} = K_R$ and $\sqrt{F_{at}^2 + F_{ct}^2} = F_R$.

Equation (3) showed that a positive increment in range R corresponded to a negative spatial frequency. Notice that Eqn. (6) lacks the negative sign of Eqn. (3) and so appears to show that a positive increment in x_r or y_r results in a positive spatial frequency. In fact, these equations are consistent. The location of the receiver or radar with respect to **P** determines whether a positive increment in x_r or y_r increases or decreases the range, and therefore decreases or increases the phase. As an example, for the receiver and scatterer locations shown in Figure 1, a positive increase in either x_r or y_r <u>de</u>creases range, so that the positive spatial frequency would be expected. If however x_r was greater than x_P (i.e., the receiver was to the right of **P**), a positive increment in x_r would increase the range and therefore decrease the phase. In this case the angle ψ would be such that $\sin\psi$ would be negative and Eqn. (6) would correctly reflect the phase decrease. The reader can see that other relative positions of the radar and **P** will result in the sign of $\sin\psi$ being such that the phase always decreases if the range is increasing, as should be expected.

One might consider the one-way *cross*-range (orthogonal to range, also sometimes called the *azimuth* dimension) spatial frequency as well. Equation (2) shows that there is no dependence of the phase on the cross-range dimension, so $K_{CR} = 0$ and $F_{CR} = 0$.

These conclusions are also easily obtained simply by observing the effective wavelength (period) in each dimension. Refer again to Figure 1. In the range dimension (direction of propagation), the wavelength is λ meters, so the expected spatial frequency in cycles/m is the inverse of this, $F_R = 1/\lambda$ cycles/m. In the along-track dimension, the geometry shows that the wavelength is $\lambda_{at} = \lambda/\sin\psi$ meters, so the corresponding spatial frequency is $F_{at} = \sin\psi/\lambda$ cycles/m, in agreement with Eqn. (5). Similarly, $\lambda_{ct} = \lambda/\cos\psi$ meters and $F_{ct} = \cos\psi/\lambda$ cycles/m as in Eqn. (6). Finally, the phase does not vary along a wavefront, so the period in the cross-range dimension is infinite and the corresponding frequency is zero: $\lambda_{cr} = \infty$, $F_{cr} = 0$ cycles/m as discussed above.

2 Two-way Spatial Frequency

In many, perhaps most cases of interest in radar, the receiver is replaced by a monostatic radar transmitter and receiver. The radar transmits the signal or interest, and **P** represents a scatterer which produces an echo back at the radar receiver. In this case, the signal travels a distance $2R$ instead of R with the result that $y(t) = A\exp\left[j\left(\Omega\left(t - 2R/c\right) + \phi\right)\right]$. The phase is then $\Phi(t,R)$ = $\left\{\Omega t - (4\pi/\lambda)R + \phi\right\}$ or $\left\{\Omega t - (4\pi/\lambda)\sqrt{\left(x_{\mathrm{P}} - x_r\right)^2 + \left(y_{\mathrm{P}} - y_r\right)^2} + \phi\right\}$. Consequently, there is no change in the temporal frequency of the observed signal, but all spatial frequencies are doubled. For example, $F_R' = -2/\lambda$ cycles/m. Here, we use the "prime" (apostrophe) to denote two-way spatial frequencies.

3 Three-Dimensional Spatial Frequency

While we have kept to two dimensions in this discussion for simplicity of equations and illustrations, it should be apparent that all of the results are easily extended to the three-dimensional case, with no new issues arising.

4 References

[1] M. A. Richards, "Virtual Arrays and coarrays, part 1: Phase Centers and Virtual Elements", technical memo, Feb. 2017. Available at http://www.radarsp.com

[2] M. A. Richards, *Fundamentals of Radar Signal Processing*, second edition. McGraw-Hill, 2014.

[3] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, third edition. Pearson, 2010.