

# Relationship Between Temporal and Spatial Frequency in Radar Imaging

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## 1 Spatial Frequency of EM Waves

Propagating electromagnetic (EM) waves of interest in radar vary sinusoidally in both time and frequency. If the radar frequency is  $F$  Hz (cycles/sec), then the temporal period observed at a fixed point in space is  $1/F = T$  seconds. Assume the propagation is in the  $y$  direction in Cartesian spatial coordinates. The wavelength is  $\lambda = c/F$  meters (or meters/cycle), where  $c$  is the speed of light. The *spatial frequency* of the same EM wave is then  $1/\lambda = F/c$  cycles/m, which we will denote as  $F_y$  (there seems to be no commonly-agreed symbol for spatial frequency in cyclical units). It is more common to consider the spatial frequency in units of radians/m,  $2\pi/\lambda = 2\pi F/c$ ; this is usually denoted with the symbol  $k_y$  and called the *wavenumber*. I prefer to use the cyclical units. I also prefer to use upper case  $K_y$  for wavenumber and reserve the lower case  $k_y$  for normalized frequencies.

## 2 Temporal Fourier Transform

Using “cyclical units” (*i.e.*, Hz instead of rads/s), the Fourier transform  $X(F)$  of a time function  $x(t)$  is given by

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt \quad (1)$$

and the inverse transform is

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{+j2\pi Ft} dF \quad (2)$$

## 3 Time-Range Equivalence and the Spatial Fourier Transform

Range  $R$  and time delay  $t$  are related in monostatic radar according to [1]

$$t = \frac{2R}{c} \quad (3)$$

Suppose  $x(t)$  is the output of a radar receiver as a function of time; we can use (3) to relabel the time axis in units of range, giving a *range profile* of the scene viewed by the

radar. We would like to have a relationship between the range profile and a spectrum expressed in spatial frequency units. Using the relationship (3), we can re-write (1) as

$$X(F) = \int_{-\infty}^{\infty} x\left(\frac{2R}{c}\right) e^{-j4\pi RF/c} \cdot \frac{2}{c} dR \quad (4)$$

Note that the quantity  $4\pi RF/c$  often appears as  $4\pi/\lambda$ , but we will leave it in terms of frequency here.

Now define

$$\tilde{x}(R) = \frac{2}{c} x\left(\frac{2R}{c}\right) \quad (5)$$

The integral in (4) is now

$$\int_{-\infty}^{\infty} \tilde{x}(R) e^{-j2\pi(2F/c)R} dR \quad (6)$$

Note that this is exactly of the form of a Fourier transform (compare to Eq. (1)) of a function of range  $R$  provided we identify the quantity  $2F/c$  as the *range frequency* variable  $F_r$ . That is,

$$\tilde{X}(F_r) = \int_{-\infty}^{\infty} \tilde{x}(R) e^{-j2\pi F_r R} dR \quad (7)$$

To emphasize, the range frequency variable  $F_r$  in  $\text{m}^{-1}$  or cycles/m is related to temporal frequency  $F$  in  $\text{s}^{-1}$  or cycles/s (Hz) according to

$$F_r \text{ m}^{-1} = \frac{2}{c} F \text{ s}^{-1} \quad (8)$$

Note that range frequency  $F_r$  is *not* the same as spatial frequency  $F_y$ . In particular,  $F_r = 2F_y$ .

## 4 Consequences

Equation (7) shows that a range profile forms a Fourier transform pair with a spectrum expressed in range frequency units. This means that we can reason about signal properties such as relationships between bandwidth and resolution, sampling and aliasing, and so forth using the same rules that apply to temporal signals and their Fourier transforms. In particular,

- Analogous to the fact that a resolution of  $\Delta t$  seconds requires a signal bandwidth of  $\Delta F = 1/\Delta t$  Hz, a range resolution of  $\Delta R$  m requires a range frequency bandwidth of  $1/\Delta R$  cycles/m. Note that the range frequency bandwidth of  $1/\Delta R$  is equivalent to  $c/2\Delta R$  Hz in temporal units, which is the usual formula relating range resolution to temporal frequency bandwidth [1].
- Sampling a range profile at intervals of  $\delta R$  m will cause its range frequency spectrum to replicate at intervals of  $1/\delta R$  cycles/m in range frequency.
- If a signal has a range frequency bandwidth of  $B_r$  cycles/m, the Nyquist theorem requires a sampling rate of at least  $B_r$  samples/m, corresponding to a sampling interval of  $1/B_r$  m, to avoid aliasing.
- If a range frequency spectrum is sampled at an interval  $\delta F_r$  in range frequency (for instance, through computation of a discrete Fourier transform), the range profile will be replicated at intervals of  $1/\delta F_r$  m in range.
- Thus, the spectrum of a range profile of length  $R_{max}$  m should be sampled in range frequency at intervals of  $1/R_{max}$  or less to avoid aliasing.

## 5 Cross-Range Spatial Frequency

In [1], Eq. 8.34, it is shown that the cross-range spatial frequency  $K_{cr}$  (denoted  $K_u$  in [1]) is given by  $(4\pi/\lambda)(x/R)$  rads/m, where  $R$  is range from the radar to the scatterer and  $x$  is the cross-range displacement of the scatterer relative to the normal to the radar flight path, as shown in Fig. 1. In cyclical units, then, the spatial frequency is  $(2/\lambda)(x/R)$  cycles/m. Note that  $x/R = \sin\theta$ , where  $\theta$  is the angle of the scatterer relative to the normal to the boresight. For small angles,  $\sin\theta \cong \theta$ . Thus, we get the conversion from angle to cross-range spatial frequency in cycles/m or  $m^{-1}$

$$F_{cr} = \frac{2}{\lambda}\theta \quad (9)$$

Spatial resolution, replication interval, and sampling can be reasoned about in terms of cross-range spatial frequency  $F_{cr}$  in the same manner as described above for range spatial frequency  $F_r$ .

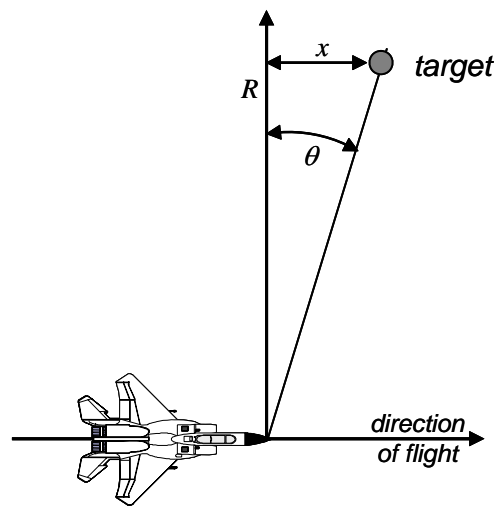


Figure 1. Geometry for cross-range spatial frequency.

## 6 References

- [1] M. A. Richards, *Fundamentals of Radar Signal Processing*. McGraw-Hill, New York, 2005.