The SNR of a Simple Pulse in Noise

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1 Acknowledgement

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2 Problem Statement

Consider a simple baseband pulse $x(t)$ of amplitude A and duration τ embedded in additive stationary white Gaussian noise (WGN) $w(t)$. The spectrum $X(F)$ of the pulse is the sinc function $A\sin(\pi F\tau)/\pi F$, which has a peak amplitude of $A\tau$ and a Rayleigh (peak-to-first-null) bandwidth of $1/\tau$ Hz. Assume the noise has a two-sided power spectral density (PSD) of N_0 W/Hz.

The signal $y(t) = x(t) + w(t)$ is passed through the matched filter for the rectangular pulse. The impulse response of the matched filter is a rectangular pulse of amplitude α and duration τ. Its frequency response $H(F) = \alpha\sin(\pi F\tau)/\pi F$ has peak amplitude $a\tau$ and Rayleigh bandwidth $1/\tau$ Hz. The purpose of this memo is to determine the effect of the matched filter on the signal-to-noise ratio (SNR) of the signal by computing the SNR before and after the matched filter operation.

3 SNR Before Matched Filtering

Signal: We define SNR as the peak signal power (voltage squared), divided by the average noise power. Before the matched filter, the signal amplitude is simply A, so the signal peak power is $S = A^2$.

Noise: To get a value for average noise power, it is necessary to define a finite bandwidth B_{obs} through which the noise is observed; otherwise, the noise power would be infinite. In order to pass the pulse with no significant distortion or loss of energy, B_{obs} must include essentially all of the energy in $x(t)$, and therefore must include "many" sidelobes of $X(F)$. The noise power viewed through this bandwidth is $N = B_{\text{obs}} N_0$.

Combining these results give the SNR prior to matched filtering as

$$SNR \text{ before matched filtering} \equiv \chi = \frac{S}{N} = \frac{A^2}{B_{\text{obs}} N_0} \tag{1}$$

4 SNR After Matched Filtering

Signal: The output of the matched filter in response to the input pulse will be a triangle of length 2τ and peak amplitude $\alpha A\tau$; see section 4.2.2 in [1] for this calculation. The post-matched filter signal peak power is therefore $\alpha^2 A^2 \tau^2 \equiv S_{\text{mf}}$.

Noise: The noise PSD at the matched filter output will be the input PSD times the "power spectrum" of the matched filter, i.e. $N_0 \cdot |H(F)|^2$. The noise power at the output is obtained by integrating this PSD over frequency and applying Parseval's theorem.

$$N_{\text{mf}} = N_0 \int_{-\infty}^{\infty} |H(F)|^2 \, dF = N_0 \int_{-\infty}^{\infty} |h(t)|^2 \, dt = N_0 \alpha^2 \tau \tag{2}$$

The SNR after matched filtering is

$$SNR \text{ after matched filtering} \equiv \chi_{\text{mf}} = \frac{S_{\text{mf}}}{N_{\text{mf}}} = \frac{A^2 \tau}{N_0} \tag{3}$$

To facilitate the "before" vs. "after" comparison of Eqs. (1) and (3), it is useful to recall the concept of *noise-equivalent bandwidth* B_{ne}. B_{ne} is the bandwidth of an ideal lowpass filter that has the same maximum power gain as $H(F)$ and the passes the same amount of white noise power to its output as does the actual system frequency response $H(F)$. B_{ne} therefore satisfies ([1], Eq. (2.78))

$$B_{\text{ne}} \max\left\{|H(F)|^2\right\} = \int_{-\infty}^{\infty} |H(F)|^2 \, dF \quad \rightarrow \quad B_{\text{ne}} \equiv \frac{\int_{-\infty}^{\infty} |H(F)|^2 \, dF}{\max\left\{|H(F)|^2\right\}} = \frac{\int_{-\infty}^{\infty} |h(t)|^2 \, dt}{\max\left\{|H(F)|^2\right\}} = \frac{\alpha^2 \tau}{\alpha^2 \tau^2} = \frac{1}{\tau} \tag{4}$$

Parseval's theorem has again been used to get the second form, which in turn has been used to compute B_{ne} for the simple pulse matched filter. Notice that the noise equivalent bandwidth equals the Rayleigh bandwidth in this example. This also means $B_{\text{obs}} \gg B_{\text{ne}}$.

The matched filter output noise power can now be expressed as $N_{\text{mf}} = N_0 B_{\text{ne}} G_{\text{mf}} = N_0 B_{\text{ne}} \alpha^2 \tau^2$, where G_{mf} is the matched filter power gain. The output SNR becomes

$$SNR \text{ after matched filtering} \equiv \chi_{\text{mf}} = \frac{S_{\text{mf}}}{N_{\text{mf}}} = \frac{A^2}{B_{\text{ne}} N_0} \tag{5}$$

Comparing Eqs. (1) and (5) shows that the matched filter has increased the SNR by the factor $B_{\text{obs}}/B_{\text{ne}}$. While this sounds like an SNR gain, it should probably *not* be thought of in those terms. The reason is that the observation bandwidth B_{obs} may not correspond to the actual bandwidth of any component of the radar system, but instead is a mathematical convenience used to define a finite input noise power.

5 Additional Comments

5.1 Discrete-Time Case

In a digital implementation of the matched filter or other receiver filter, assume the simple pulse is represented by M samples over its length of τ seconds. That is, the sampling interval is $T_s = \tau/M$ seconds and the sampling frequency is $F_s = 1/T_s = M/\tau$ samples/second, which is M times the nominal pulse bandwidth (and therefore Nyquist sampling rate) of $1/\tau$ Hz again. This implies that the analog signal has been bandpass filtered to a bandwidth of M/τ Hz prior to the sampling operation (analog-to-digital conversion, ADC) to avoid aliasing. Effectively, $B_{obs} = M/\tau$ Hz. The increase in SNR from the output of the pre-sampling filter to the output of the matched filter is then $B_{obs}/B_{ne} = M$. This increase might reasonably be considered a gain due to the matched filter relative to the SNR that existed at the ADC output.

5.2 Simple Pulse vs. Pulse Compression Waveforms

It is shown in [1] that, in general, the peak SNR at the output of a filter matched to a particular waveform is E/N_0, where E is the waveform energy. For the simple pulse $E = A^2\tau$, so that Eq. (3) is consistent with this claim. Also recall that when the matched filter is used, the time resolution at the output equals $1/B$ Hz, where B is the waveform bandwidth [1].

The rectangular pulse has the property that its time-bandwidth product (TBP, waveform duration times waveform bandwidth) equals 1, assuming the bandwidth is taken as $1/\tau$ Hz. That choice of bandwidth is somewhat arbitrary, since a rectangular pulse is not strictly bandlimited. However, it is in common use. It equals the 4 dB bandwidth, the Rayleigh bandwidth, and (as we have seen) the noise-equivalent bandwidth of the pulse. On the other hand, it captures only 78% of the total energy in the pulse power spectrum $|X(F)|^2$.

Pulse compression waveforms have TBPs greater than 1, often much greater. Consider a waveform of amplitude A, duration τ seconds, and bandwidth $B = k(1/\tau),\ k \gg 1$ Hz so that the TBP = $k \gg 1$. This waveform has the same energy as the simple pulse used above, and so will exhibit the same peak SNR at the output of its matched filter.[1] However, the pulse compression waveform will have a time (thus range) resolution that is $1/k$ that of the simple pulse because of its wider bandwidth. Therefore, the use of pulse compression provides a factor of k improvement in range resolution relative to a simple pulse providing the same SNR. Alternatively, a simple pulse having the same range resolution as the pulse compression waveform will have to be shorter by a factor of k, i.e. a duration of τ/k seconds. This will reduce its energy and therefore SNR at the matched filter output by a factor of k. In this comparison, pulse compression produces an SNR gain of a factor of k compared to a simple pulse of the same range resolution.

[1] Each waveform is processed through its own matched filter, which are not the same since the waveforms aren't the same.

5.3 Bandpass Receiver Filter

Many radar systems, particularly older or simpler ones lacking modern digital receivers, may have a frequency response that is approximately an ideal bandpass filter (BPF) rather than a true matched filter. This raises the question of how one should choose the BPF bandwidth to maximize the output SNR, and how much less that SNR is than the maximum SNR provided by the matched filter. This question is addressed in [2], which repeats and discusses an analysis from 1963. In short, the answer is that the maximum SNR occurs when the two-sided BPF bandwidth equals $1.37/\tau$ Hz (cutoff frequency = $0.685/\tau$ Hz). At that bandwidth, the SNR at the filter output is 82% of the matched filter SNR, a loss of 0.84 dB.

6 References

[1] M. A. Richards, *Fundamentals of Radar Signal Processing*, second edition. McGraw-Hill, 2014.

[2] M. A. Richards, "Optimum Bandpass Filter Bandwidth for a Rectangular Pulse", technical memo, July 2015. Available at http://www.radarsp.com