revised 2/8/2010

Rice Distribution for RCS

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September 2, 2006

1 The Rice Power Distribution in *Fundamentals of Radar Signal Processing*

The Rice or Rician distribution for RCS σ in the form given in [1] is

$$p_\sigma(\sigma) = \frac{1}{\bar{\sigma}}\left(1+a^2\right)\exp\left[-a^2 - \frac{\sigma}{\bar{\sigma}}\left(1+a^2\right)\right] I_0\left[2a\sqrt{\left(1+a^2\right)\left(\sigma/\bar{\sigma}\right)}\right] \tag{1}$$

where $\bar{\sigma}$ is the mean value of σ. This pdf describes the power of the received echo signal when the target consists of "many" small scatterers of approximately equal strength, plus one "dominant scatterer" whose individual RCS equals that of all the small scatterers combined.

Call the small scatterers the "Rayleigh scatterers" (because if we had only those, the pdf of the echo voltage would be Rayleigh and the pdf of the echo power would be exponential). The mean RCS of the echo from just the Rayleigh scatterers is denoted $\bar{\sigma}_R$. The RCS of the dominant scatterers is then $a^2\bar{\sigma}_R$, and the mean RCS of all of the scatterers is $\bar{\sigma} = \left(1+a^2\right)\bar{\sigma}_R$. The pdf of Eq. (1) was obtained from Eq. 3-91 of [2], where it is given in the form

$$p_x(x) = \frac{1}{\psi_0}\exp\left[-s - \frac{x}{\psi_0}\right] I_0\left[2a\sqrt{sx/\psi_0}\right] \tag{2}$$

The notation changes $s \rightarrow a^2$, $\psi_0 \rightarrow \bar{\sigma}_R$, in combination with the equivalence $\bar{\sigma} = \left(1+a^2\right)\bar{\sigma}_R$ above, convert Eq. (2) to the form of Eq. (1). The reason I prefer the form of Eq. (1) is that it uses the mean of the complete dominant+Rayleigh scatterers signal $\bar{\sigma}$, rather than the mean of the only Rayleigh component, $\bar{\sigma}_R$.

Note that in the absence of a dominant scatterer, $a^2 = 0$, Eq. (1) reduces to the exponential pdf, as should be expected. (This result uses the fact that $I_0(0) = 1$.)

Note also that Eqns. (1) or (2) are more properly called non-central chi-square distributions. See Section 4 of this note for a comment on this terminology.

2 The Rice Voltage Distribution in *Fundamentals of Radar Signal Processing*

Equation (1) is the pdf of the received power σ. The pdf of the magnitude of the received voltage, $\varsigma = \sqrt{\sigma}$, can be found using standard results found in most random variable textbooks, *e.g.* [3]. Specifically, for the transformation $\varsigma = \sqrt{\sigma}$,

$$p_\varsigma(\varsigma) = \frac{p_\sigma\left(\varsigma^2\right)}{d\varsigma/d\sigma} = 2\varsigma p_\sigma\left(\varsigma^2\right) \tag{3}$$

Using Eq. (1) in (3) immediately gives the Rician distribution for voltage in the form given in [1]:

$$p_\varsigma(\varsigma) = \frac{2\varsigma\left(1+a^2\right)}{\bar{\sigma}} \exp\left[-a^2 - \frac{\varsigma^2}{\bar{\sigma}}\left(1+a^2\right)\right] I_0\left(2a\varsigma\sqrt{\left(1+a^2\right)/\bar{\sigma}}\right) \tag{4}$$

Note that if $a^2 = 0$, this reduces to a Rayleigh distribution, as should be expected.

3 "The Rice Distribution"

The form usually cited for the Rice distribution is [4]:

$$p_z(z) = \frac{z}{\chi^2} \exp\left[\frac{-z^2 - \alpha^2}{2\chi^2}\right] I_0\left(\frac{\alpha z}{\chi^2}\right) \tag{5}$$

How does this relate to either Eq. (1) or (2)? First, the linear term in z means that it has to be compared to the voltage distribution of Eq. (4), since there is no linear term in σ in Eq. (1). If we make the identifications

$$\begin{aligned} z &\to \varsigma \\ \alpha^2 &\to \left(\frac{a^2}{1+a^2}\right)\bar{\sigma} \\ \chi^2 &\to \frac{\bar{\sigma}}{2\left(1+a^2\right)} \end{aligned}$$

then Eq. (5) becomes identical to Eq. (4). Thus the Rician voltage distribution of Eq. (4) is consistent with the most common definition of the Rice distribution. From these transformations and earlier equation it is easy to see that $\alpha^2 = a^2\bar{\sigma}_R$, which is the dominant scatterer RCS, and that $2\chi^2$ is the RCS of the Rayleigh component (small scatterers only), $\bar{\sigma}_R$.

4 The Unfortunate Tendency in Radar to Call Power Distributions by the Name of the Voltage Distribution

As mentioned above, Eqns. (1) and (2) are non-central chi-square distributions; see [5], for instance. It is nonetheless common in the radar community to refer to these as Rice or Rician distributions; see [2], for instance. This is the result of applying the correct name for a voltage distribution to the distribution for power that results when the voltage variable is squared. The most common example of this is references to "Rayleigh power" or "Rayleigh radar cross section" (RCS). To this author's knowledge, the Rayleigh distribution is not used to model RCS. The Rayleigh PDF is the correct model for complex Gaussian noise voltage and also for the echo voltage of a target composed of "many" unresolved scatterers of approximately equal individual RCS. When a Rayleigh-distributed random variable x representing a voltage is squared to give a new random variable y representing the corresponding power, the actual PDF of y will be exponential, not Rayleigh. Nonetheless, the literature frequently refers to a Rayleigh model for RCS or for noise power. The reader must be careful to realize that such a reference probably means Rayleigh voltage, but exponential power. The statement above and in [1] that Eqn. (1) is a Rician distribution for RCS is another example of this unfortunate tradition. (I promise to do better in the future.)

5 References

[1] M. A. Richards, *Fundamentals of Radar Signal Processing*. McGraw-Hill, New York, 2005.

[2] D. P. Meyer and H. A. Mayer, *Radar Target Detection: Handbook of Theory and Practice*. Academic Press, New York, 1973.

[3] A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes*. McGraw-Hill, New York, 2001.

[4] For example, see http://mathworld.wolfram.com/RiceDistribution.html.

[5] S. M. Kay, *Fundamentals of Statistical Signal Processing Vol. II, Detection Theory*. Prentice-Hall, New Jersey, USA, 1998.