Resampling by DFT Zero Padding

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Problem Statement

Reference [1] discusses the use of zero padding of discrete Fourier transforms (DFTs) as a means of interpolating the signal to a higher sampling rate. Only integer upsampling ratios are considered. In this note we generalize that result, considering an increase of the DFT size from K_1 to any new size $K_2 > K_1$. We also show that the technique implies interpolation of new samples with a Dirichlet function, often called an "aliased sinc" (asinc) or "digital sinc" (dsinc) function, as might be expected for bandlimited interpolation.

Let $x_1[n]$ be a sequence of length N. Let $X_1[k]$ be the K_1-point DFT of x_1, $K_1 \geq N$. Assume that K_1 is odd; we will deal with the even K_1 case later. Form a new, larger K_2-point DFT, $K_2 > K_1$, using the technique described in [1]. Finally, compute the inverse DFT of $X_2[k]$ to obtain the new sequence $x_2[n]$. Our goal is to determine how x_2 is related to x_1.

The Case of K_1 Being Odd

By definition of the inverse DFT, we have

$$x_2[n] = \frac{1}{K_2} \sum_{k=0}^{K_2-1} X_2[k] e^{j2\pi nk/K_2}, \quad n = 0, \ldots, K_2 - 1 \tag{1}$$

Using the relationship between X_2 and X_1 defined in [1], this becomes

$$\begin{aligned} x_2[n] &= \frac{1}{K_2} \left\{ \sum_{k=0}^{(K_1-1)/2} \frac{K_2}{K_1} X_1[k] e^{j2\pi nk/K_2} + \sum_{k=(K_1+1)/2}^{K_1-1} \frac{K_2}{K_1} X_1[k] e^{j2\pi n(k+K_2-K_1)/K_2} \right\} \\ &= \frac{1}{K_1} \left\{ \sum_{k=0}^{(K_1-1)/2} X_1[k] e^{j2\pi nk/K_2} + \sum_{k=(K_1+1)/2}^{K_1-1} X_1[k] e^{j2\pi n(k-K_1)/K_2} \right\} \end{aligned} \tag{2}$$

where we have used the fact that $\exp(j2\pi nK_2/K_2) = 1$ in the last step. Now perform a change of variable $k' = k - K_1$ in the second summation, giving

$$
\begin{aligned}
x_2[n] &= \frac{1}{K_1}\left\{ \sum_{k=0}^{(K_1-1)/2} X_1[k]e^{j2\pi nk/K_2} + \sum_{k=-(K_1-1)/2}^{-1} X_1[k+K_1]e^{j2\pi nk/K_2} \right\} \\
&= \frac{1}{K_1}\left\{ \sum_{k=0}^{(K_1-1)/2} X_1[k]e^{j2\pi nk/K_2} + \sum_{k=-(K_1-1)/2}^{-1} X_1[k]e^{j2\pi nk/K_2} \right\} \\
&= \frac{1}{K_1} \sum_{k=-(K_1-1)/2}^{(K_1-1)/2} X_1[k]e^{j2\pi nk/K_2} \qquad (K_1 \text{ odd})
\end{aligned} \tag{3}
$$

The second line used the fact that $X_1[k]$ has period K_1. Define $\alpha = (K_1-1)/2$ for brevity, insert the definition of X_1 as the DFT of x_1, and interchange summations:

$$
\begin{aligned}
x_2[n] &= \frac{1}{K_1} \sum_{k=-\alpha}^{\alpha} \left(\sum_{m=0}^{N-1} x_1[m]\, e^{-j2\pi nk/K_1} \right) e^{j2\pi nk/K_2} \\
&= \frac{1}{K_1} \sum_{m=0}^{N-1} x_1[m] \sum_{k=-\alpha}^{\alpha} \exp\left[j2\pi k \left(\frac{n}{K_2} - \frac{m}{K_1} \right) \right] \\
&= \frac{1}{K_1} \sum_{m=0}^{N-1} x_1[m] \sum_{k=-\alpha}^{\alpha} \exp(j\beta k) \\
&= \frac{1}{K_1} \sum_{m=0}^{N-1} x_1[m]\, Q(n,m;K_1,K_2)
\end{aligned} \tag{4}
$$

where we have defined $\beta = 2\pi(n/K_2 - m/K_1)$, again for brevity. The second summation is the interpolating function that defines the weights by which samples of x_1 are combined to form samples of x_2. We denote this function as $Q(n,m;K_1,K_2)$. Q is a function of the two variables n and m, and has two parameters, K_1 and K_2.

Using the geometric sum formula, we can get a closed form for Q:

$$
\begin{aligned}
Q(n,m;K_1,K_2) &= \sum_{k=-\alpha}^{\alpha} e^{j\beta k} = \sum_{k=0}^{2\alpha} e^{j\beta(k-\alpha)} = e^{-j\alpha\beta} \sum_{k=0}^{2\alpha} e^{j\beta k} = e^{-j\alpha\beta} \left\{ \frac{1-e^{j(2\alpha+1)\beta}}{1-e^{j\beta}} \right\} \\
&= e^{-j\alpha\beta} \left\{ e^{+j\alpha\beta} \cdot \frac{\sin[(\beta/2)(2\alpha+1)]}{\sin(\beta/2)} \right\} \\
&= \frac{\sin(\beta K_1/2)}{\sin(\beta/2)} \qquad (K_1 \text{ odd})
\end{aligned} \tag{5}
$$

Using L'Hospital's rule, it is easy to establish that the peak value of Q, which occurs when $\beta = 0$, is K_1. Also note that $\beta K_1/2 = \pi((K_1/K_2)n - m)$.

Equations (4) and (5) (and Eq. (9), yet to come) are the principal results of this note. They show that zero-padding the DFT of a sequence x_1 produces a longer sequence x_2 which is obtained by Dirichlet interpolation of the values of x_1. At appropriate sample locations, x_2 exactly matches x_1. The case of upsampling by an integer factor D that was considered in [1] is a special case of the analysis here with $K_2 = D \cdot K_2$.

The intent of this procedure is to increase the sampling rate by the ratio K_2/K_1. As a result, we would expect $x_2[n]$ to equal $x_1[m]$ whenever $n = (K_2/K_1)m$. This will be the case if, for a given value of n in Eq. (4), Q takes on the value K_1 for any m such that $n = (K_2/K_1)m$, and a value of zero for all other m. To see that this property is satisfied, substitute $(K_2/K_1)m$ for n in the definition of β. The result is that $\beta = 0$ so that $Q = K_1$ as desired. All other values of m for a given n can be considered to be of the form $m = (K_1/K_2)n - l$ for some integer $l \neq 0$.In this case $\beta = 2\pi l/K_1$. Then the $\sin(\beta K_1/2)$ term of Q is zero, while the $\sin(\beta/2)$ term is not. Thus, Q is zero for these values of m.

If l is an integer multiple of K_1, Q would again be nonzero. However, x_1 is nonzero only for $0 \leq N-1 \leq K_1-1$, so $x_1[m]$ is itself zero for the resulting values of m and these values of m are not of concern.

The Case of K_1 Being Even

The approach and many of the details for the even K_1 case are the same as for the odd case. The difference is in the details of the interpolating function Q. Referring again to [1] for the manner in which the samples of X_1 are distributed into X_2, the equivalent of the key steps in Eqs. (2) and (3) is

$$
\begin{aligned}
x_2[n] &= \frac{1}{K_2}\left\{\sum_{k=0}^{K_1/2-1}\frac{K_2}{K_1}X_1[k]e^{j2\pi nk/K_2} + \frac{1}{2}\frac{K_2}{K_1}X_1\left[\frac{K_1}{2}\right]e^{j2\pi nK_1/2K_2} + \dots\right. \\
&\qquad \left.\dots + \frac{1}{2}\frac{K_2}{K_1}X_1\left[\frac{K_1}{2}\right]e^{j2\pi n(K_2-K_1/2)/K_2} + \sum_{k=K_1/2+1}^{K_1-1}\frac{K_2}{K_1}X_1[k]e^{j2\pi n(k+K_2-K_1)/K_2}\right. \\
&= \frac{1}{K_1}\left\{\sum_{k=0}^{K_1/2-1}X_1[k]e^{j2\pi nk/K_2} + \frac{1}{2}X_1\left[\frac{K_1}{2}\right]e^{j\pi nK_1/K_2} + \dots\right. \\
&\qquad \left.\dots + \frac{1}{2}X_1\left[\frac{K_1}{2}\right]e^{-j\pi nK_1/K_2} + \sum_{k=K_1/2+1}^{K_1-1}X_1[k]e^{j2\pi n(k-K_1)/K_2}\right. \\
&= \frac{1}{K_1}\sum_{k=-(K_1/2-1)}^{K_1/2-1}X_1[k]e^{j2\pi nk/K_2} + X_1\left[\frac{K_1}{2}\right]\cos\left(\frac{K_1}{K_2}n\pi\right) \qquad (K_1 \text{ even})
\end{aligned}
\tag{6}
$$

The two middle terms in the first line of this equation are a result of the way the assignment of values from X_1 to X_2 is made, specifically the splitting of the middle sample of X_1 in two, as described in [1]. Defining $\gamma = K_1/2-1$, keeping β as before, and inserting the definition of $X_1[k]$, the equivalent of Eq. (4) is

$$
\begin{aligned}
x_2[n] &= \frac{1}{K_1}\sum_{m=0}^{N-1} x_1[m]\left\{(-1)^m \cos\left(\frac{K_1}{K_2}n\pi\right) + \sum_{k=-\gamma}^{\gamma}\exp(j\beta k)\right\} \\
&= \frac{1}{K_1}\sum_{m=0}^{N-1} x_1[m]\,Q(n,m;K_1,K_2)
\end{aligned} \tag{7}
$$

The $(-1)^m$ term arose from the definition of $X_1[K_1/2]$. The summation term in Q can be put in closed form as

$$
\begin{aligned}
\sum_{k=-\gamma}^{\gamma} e^{j\beta k} &= \sum_{k=0}^{2\gamma} e^{j\beta(k-\gamma)} = e^{-j\gamma\beta}\sum_{k=0}^{2\gamma} e^{j\beta k} = e^{-j\gamma\beta}\left\{\frac{1-e^{j(2\gamma+1)\beta}}{1-e^{j\beta}}\right\} \\
&= e^{-j\gamma\beta}\cdot\left\{e^{+j\gamma\beta}\cdot\frac{\sin\left[(\beta/2)(2\gamma+1)\right]}{\sin(\beta/2)}\right\} \\
&= \frac{\sin\left[\beta(K_1-1)/2\right]}{\sin(\beta/2)}
\end{aligned} \tag{8}
$$

Thus, the interpolating function Q becomes

$$
Q(n,m;K_1,K_2) = (-1)^m\cos\left(\frac{K_1}{K_2}n\pi\right) + \frac{\sin\left[\beta(K_1-1)/2\right]}{\sin(\beta/2)} \qquad (K_1 \text{ even}) \tag{9}
$$

For the sanity check, consider the case where $n = (K_2/K_1)m$ again. It is easy to see from (9) that the first term will be $(-1)^m\cos(m\pi)=1$. Since $\beta = 0$ the second term reduces to K_1-1, so that in total $Q = K_1$, as desired. For the case where $m = (K_1/K_2)n-l$ for some integer $l \neq 0$ so that $\beta = 2\pi l/K_1$, the first term of Q becomes $(-1)^l$. The “sinc” portion of the second term becomes

$$
\frac{\sin\left[\dfrac{2\pi l(K_1-1)}{2K_1}\right]}{\sin\left(\dfrac{2\pi l}{2K_1}\right)} = \frac{\sin\left(\pi l - \dfrac{\pi l}{K_1}\right)}{\sin\left(\dfrac{\pi l}{K_1}\right)} = -\frac{\sin\left(\dfrac{\pi l}{K_1} - \pi l\right)}{\sin\left(\dfrac{\pi l}{K_1}\right)} = -(-1)^l\frac{\sin\left(\dfrac{\pi l}{K_1}\right)}{\sin\left(\dfrac{\pi l}{K_1}\right)} = (-1)^{l+1} \tag{10}
$$

In total, this gives $Q = 0$ as desired for these cases.

Sampling Rate and Bandwidth Interpretation

The DFT zero-padding process begins with a sequence of length K_1 (possibly zero-padded from a shorter length N) and produces a new sequence of length K_2. Suppose the original sequence $x_1[n]$ was obtained by sampling a continuous-time signal $x_1(t)$ at a sampling interval of T_1 seconds. It then follows that the DTFT $X_1(\omega)$ covers a frequency range of width $F_{s1} = 1/T_1$ Hz. The DFT $X_1[k]$ samples $X_1(\omega)$ every F_{s1}/K_1 Hz. But what is the effective sampling rate of $x_2[n]$?

There is more than one reasonable interpretation. The most common use of this technique assumes that the DFT sample spacing is held constant, so that zero-padding from K_1 to K_2 frequency samples increases the bandwidth represented by the DFT by the factor K_2/K_1. This in turn implies a new time-domain sampling rate of $F_{s2} = (K_2/K_1)F_{s1}$ and a corresponding new sampling interval of $T_2 = (K_1/K_2)T_1$. The total duration of $x_2[n]$ in seconds is then the same as that of $x_1[n]$, so the result is that x_1 has been upsampled to a higher sampling rate. This viewpoint is the interpretation commonly applied to the use of DFT zero padding.

One could just as easily assume that the sequence $x_2[n]$ retains the same original sampling rate of T_1. In this case, the total bandwidth represented in X_2 must still be F_{s1} Hz. This implies that the zero padding of the DFT has compressed the original spectral data in X_1 into a frequency interval that is smaller by the factor (K_1/K_2). The time sequence x_2 is now longer in seconds than x_1 by the factor (K_2/K_1). Under this interpretation, the process is viewed as stretching the time base of the signal.

References

[1] M. A. Richards, "A Note on Upsampling by Integer Factors Using the DFT", technical memo, February 24, 2012. Available at www.radarsp.com.