Derivation of the Range-Doppler Algorithm Frequency Response

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Based closely on notes developed by
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1 Introduction

Equation (8.43) in [1] gives the impulse response of a stripmap synthetic aperture radar (SAR) system to a point scatterer at downrange position R and cross-range position x relative to the center of the synthetic aperture. To derive the range-Doppler SAR image formation algorithm, it is assumed that the swath length of the image is small compared to the nominal range. That is, if R is expressed as $R = R_0+\delta R$ where R_0 is the center of the range swath, then $|\delta R_{max}/R_0| << 1$. In this case, the range migration function during the data collection can be simplified to the form given in Eq. (8.45) of [1], which is

$$R(u;x) \cong \sqrt{(u-x)^2 + R_0^2} + \delta R \tag{1}$$

where u is the position of the radar sensor in the cross-range dimension.

Reference [1] presents the frequency response of the SAR system under this short swath approximation. The result is given in Eq. (8.50), which is repeated here:

$$H(K_u, \Omega; R_0) = \sqrt{\frac{\pi c R_0}{j\Omega}} \exp\left\{ + jR_0 \left[\frac{2\Omega}{c} - \sqrt{\left(\frac{2\Omega}{c}\right)^2 - K_u^2} \right] \right\} \tag{2}$$

In this equation, K_u is the cross-range spatial frequency in radians/meter and Ω is the downrange temporal frequency in radians/second. This result assumes that pulse compression has already been performed; more discussion of this point is below.

The derivation of (2), which relies on the Principle of Stationary Phase [2]-[5] as well as additional assumptions, is tedious and lengthy and so is not described in [1]. The purpose of this note is to provide the detailed derivation. This derivation follows exactly that given by Dr. Gregory A. Showman of the Georgia Tech Research Institute in [6]. Notation is modified to follow that of [1].

2 Waveforms and the SAR System Impulse Response

Before deriving the SAR frequency response, it is useful to consider the role of the specific waveform used. The frequency response of Eq. (2) does not assume any specific radar waveform. Thus, it is applicable to a wide variety of waveforms, from conventional narrowband modulated or unmodulated pulses to unconventional ultrawideband waveforms.

For a given waveform $x(t)$, the raw received data $y(t)$ from a scatterer at range $R(u)$ when the radar is at position u will be simply the appropriately time-delayed waveform

$$y(u,t) = x\left(t - 2R(u)/c\right) \tag{3}$$

Pulse compression in fast time (range) will convolve $y(u,t)$ with the matched filter impulse response $x^*(-t)$. The matched filter output in response to the transmitted waveform $x(t)$ is $x(t) * x^*(-t) \equiv r_x(t)$. The pulse-compressed SAR data $z(u,t)$ is therefore of the form

$$z(u,t) = x^*(-t) * x\left(t - 2R(u)/c\right) = r_x\left(t - 2R(u)/c\right) \tag{4}$$

While we seek a result that is independent of any particular waveform, we will assume that the waveform supports fine resolution imaging. Thus, the matched filter output $r_x(t)$ should have a narrow mainlobe and low sidelobes. Suitable waveforms include wideband linear FM chirps, extremely short pulses, and long polyphase codes, among others. Consequently, we assume that the range profile of the matched filter output for a scatterer at range R is reasonably well-modeled as $r_x(t) = \delta_D(t-2R/c)$, where $\delta_D(\cdot)$ is the Dirac impulse function. This effectively assumes infinitely fine range resolution and ignores sidelobes at the matched filter output, but is a reasonable simplifying assumption for analyzing the cross-range signal behavior. Applying this assumption to Eq. (4), it then follows that the impulse response of the SAR data collection system for the scatterer at coordinates (x,R) is given by Eq. (8.46) of [1]:

$$h(u,t;x,R_0) = \delta_D\left(t - \frac{2}{c}R(u)\right) = \delta_D\left[\left(t - \frac{2}{c}\delta R\right) - \frac{2}{c}\sqrt{(u-x)^2 + R_0^2}\right] \tag{5}$$

Note that $h(u,t;x,R_0)$ is shift-invariant in the cross-range dimension u; that is, locating the scatterer at different values of x does not change the shape of h, but merely shifts it along the u axis. Consequently, it is adequate to consider the case of a scatterer at $x = 0$, thus eliminating one parameter. The impulse response is also shift-invariant in the relative range δR, so it may be similarly set to zero. The impulse response of interest then reduces to

$$h(u,t;R_0) = \delta_D\left[t - \frac{2}{c}\sqrt{u^2 + R_0^2}\right] \tag{6}$$

Furthermore, it is convenient to redefine the time scale of the data so that the delay to the swath center occurs at $t = 0$. This is done via the substitution $t' = t+2R_0/c$. With this substitution (and renaming t' back to t), the impulse response of interest becomes

$$h(u,t;R_0) = \delta_D\left[t + \frac{2R_0}{c} - \frac{2}{c}\sqrt{u^2 + R_0^2}\right] \tag{7}$$

This is the function for which we seek the two-dimensional Fourier transform.

3 The Range-Doppler Algorithm Frequency Response

The desired result is the 2D Fourier transform of Eq. (7), which is defined as

$$H(K_u,\Omega;R_0) \equiv \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\delta_D\left(t + \frac{2}{c}R_0 - \frac{2}{c}\sqrt{u^2 + R_0^2}\right)e^{-jK_u u}du\right)e^{-j\Omega t}dt \tag{8}$$

Start by interchanging the order of integration:

$$H(K_u,\Omega;R_0) \equiv \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\delta_D\left(t + \frac{2}{c}R_0 - \frac{2}{c}\sqrt{u^2 + R_0^2}\right)e^{-j\Omega t}dt\right)e^{-jK_u u}du \tag{9}$$

The inner integral is trivial due to the Dirac impulse function; evaluating this gives

$$\begin{aligned} H(K_u,\Omega;R_0) &\equiv \int_{-\infty}^{\infty}\exp\left[j\Omega\left(+\frac{2R_0}{c} - \frac{2}{c}\sqrt{u^2 + R_0^2}\right)\right]e^{-jK_u u}du \\ &= \exp\left(+j\Omega\frac{2R_0}{c}\right)\int_{-\infty}^{\infty}\exp\left[j\left(-\frac{2\Omega}{c}\sqrt{u^2 + R_0^2} - K_u u\right)\right]du \end{aligned} \tag{10}$$

Notice that the $\exp(-j2\Omega R_0/c)$ term is the familiar $\exp(-j4\pi R_0/\lambda)$ phase shift always observed for echo from a range R_0.

The remaining integral is approximated using the Principle of Stationary Phase (PSP). The particular form of the PSP used here was given in Eq. (4.86) of [1].[1] Consider a complex function $x(u) = A(u)\exp\left[j\theta(u)\right]$, where $A(u)$ is a real-valued envelope function and $\theta(u)$ is a complicated phase function. The one-dimensional Fourier transform of $x(u)$ is

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[1] As repeated here, the PSP equation includes correction of typographical errors that appears in the first through third printings (at least) of [1]. This and other known errata are available at www.radarsp.com.

$$
\begin{aligned}
X(\Omega) &= \int_{-\infty}^{+\infty} \underbrace{A(u)e^{j\theta(u)}}_{x(u)} e^{-j\Omega u} du = \int_{-\infty}^{+\infty} A(u)e^{j[\theta(u)-\Omega u]}du \\
&\equiv \int_{-\infty}^{+\infty} A(u)e^{j\phi(u,\Omega)}du
\end{aligned} \tag{11}
$$

The PSP then states that this integral is approximately

$$
X(\Omega) \approx \sqrt{\frac{-2\pi}{\phi''(u_0,\Omega)}}\, e^{-j\frac{\pi}{4}}\, A(u_0)e^{j\phi(u_0,\Omega)} \tag{12}
$$

where u_0 is the stationary point of ϕ, *i.e.* the value of u such that $\phi'(u_0,\Omega) = 0$. If there are multiple solutions, then $X(\Omega)$ is the sum of terms like that of Eq. (12), one for each stationary point.

To apply the PSP to the integral in Eq. (10), the phase function $\phi(t,\Omega)$ must be identified. By inspection, this is

$$
\phi(u,\Omega) = -\frac{2\Omega}{c}\sqrt{u^2+R_0^2} - K_u u \tag{13}
$$

Also note that $A(u) = 1$. The two derivatives of ϕ with respect to u are

$$
\begin{aligned}
\phi'(u,\Omega) &= \frac{d}{du}\left(-\frac{2\Omega}{c}\sqrt{u^2+R_0^2} - K_u u\right) = -\left(\frac{2\Omega}{c}\left(\frac{1}{2}\right)\frac{2u}{\sqrt{u^2+R_0^2}} + K_u\right) \\
&= -\left(\frac{2\Omega}{c}u\left(u^2+R_0^2\right)^{-1/2} + K_u\right)
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
\phi''(u,\Omega) &= -\frac{d}{du}\left(\frac{2\Omega}{c}u\left(u^2+R_0^2\right)^{-1/2} + K_u\right) \\
&= -\left(\frac{2\Omega}{c}\left(u^2+R_0^2\right)^{-1/2} + \frac{2\Omega}{c}u\left(-\frac{1}{2}2u\right)\left(u^2+R_0^2\right)^{-3/2}\right) \\
&= \frac{2\Omega}{c}u^2\left(u^2+R_0^2\right)^{-3/2} - \frac{2\Omega}{c}\left(u^2+R_0^2\right)^{-1/2}
\end{aligned} \tag{15}
$$

The stationary point(s) are the solutions of the equation $\phi'(u,\Omega) = 0$. Denoting these points as u_0, they must satisfy

$$
\phi'(u_0,\Omega) = -\left(\frac{2\Omega}{c}u_0\left(u_0^2+R_0^2\right)^{-1/2} + K_u\right) = 0 \tag{16}
$$

This implies that

$$u_0 = -\left(u_0^2 + R_0^2\right)^{1/2} \frac{c}{2\Omega} K_u \quad \Rightarrow$$

$$u_0^2 = u_0^2 \frac{c^2}{4\Omega^2} K_u^2 + R_0^2 \frac{c^2}{4\Omega^2} K_u^2 \quad \Rightarrow$$

$$u_0^2 \left(1 - \frac{c^2}{4\Omega^2} K_u^2\right) = R_0^2 \frac{c^2}{4\Omega^2} K_u^2$$

so that

$$u_0 = \pm R_0 \sqrt{\frac{\frac{c^2}{4\Omega^2} K_u^2}{1 - \frac{c^2}{4\Omega^2} K_u^2}} \tag{17}$$

A fine detail of the stationary points can be clarified by substituting this solution back into the expression for $\phi'(u)$:

$$\begin{aligned}
\phi'(u_0, \Omega) &= -\left(\frac{2\Omega}{c} u_0 \left(u_0^2 + R_0^2\right)^{-1/2} + K_u\right) \\
&= -\left(\pm \frac{2\Omega}{c} R_0 \sqrt{\frac{\frac{c^2}{4\Omega^2} K_u^2}{1 - \frac{c^2}{4\Omega^2} K_u^2}} \left(R_0^2 \frac{\frac{c^2}{4\Omega^2} K_u^2}{1 - \frac{c^2}{4\Omega^2} K_u^2} + R_0^2\right)^{-1/2} + K_u\right) \\
&= -\left(\pm \left|K_u\right| + K_u\right)
\end{aligned} \tag{18}$$

In order to make this result equal zero, the sign of u_0 must be chosen opposite to that of K_u. Thus, the stationary points are

$$u_0 = -\mathrm{sign}(K_u) R_0 \sqrt{\frac{\frac{c^2}{4\Omega^2} K_u^2}{1 - \frac{c^2}{4\Omega^2} K_u^2}} \tag{19}$$

Equation (19) can now be used to find $\phi'(u_0, \Omega)$ and $\phi''(u_0, \Omega)$, both needed for the PSP. $\phi'(u_0, \Omega)$ is found as follows:

$$\phi(u_0,\Omega) = -\frac{2\Omega}{c}\sqrt{u_0^2+R_0^2} - K_u u_0$$

$$= -\frac{2\Omega}{c}\sqrt{R_0^2\frac{\frac{c^2}{4\Omega^2}K_u^2}{1-\frac{c^2}{4\Omega^2}K_u^2}+R_0^2} + K_u\,\mathrm{sign}(k_u)R_0\sqrt{\frac{\frac{c^2}{4\Omega^2}K_u^2}{1-\frac{c^2}{4\Omega^2}K_u^2}}$$

$$= -\frac{2\Omega}{c}R_0\frac{1}{\sqrt{1-\frac{c^2}{4\Omega^2}K_u^2}} + K_u\,\mathrm{sign}(K_u)R_0\frac{\frac{c}{4\Omega}}{\sqrt{1-\frac{c^2}{4\Omega^2}K_u^2}}$$

$$= \frac{-\frac{2\Omega}{c}R_0\left(1-\frac{c^2}{4\Omega^2}K_u^2\right)}{\sqrt{1-\frac{c^2}{4\Omega^2}K_u^2}} = -\frac{2\Omega}{c}R_0\sqrt{1-\frac{c^2}{4\Omega^2}K_u^2}$$

so that finally

$$\phi(u_0,\Omega) = -R_0\sqrt{\frac{4\Omega^2}{c^2}-K_u^2} \qquad (20)$$

The curvature at the stationary points, $\phi''(u_0,\Omega)$, can be found as follows

$$\phi''(u_0,\Omega) = \frac{2\Omega}{c}u_0^2\left(u_0^2+R_0^2\right)^{-3/2} - \frac{2\Omega}{c}\left(u_0^2+R_0^2\right)^{-1/2}$$

$$= \frac{2\Omega}{c}R_0^2\frac{\frac{c^2}{4\Omega^2}K_u^2}{1-\frac{c^2}{4\Omega^2}K_u^2}\left(R_0^2\frac{\frac{c^2}{4\Omega^2}K_u^2}{1-\frac{c^2}{4\Omega^2}K_u^2}+R_0^2\right)^{-3/2} - \frac{2\Omega}{c}\left(R_0^2\frac{\frac{c^2}{4\Omega^2}K_u^2}{1-\frac{c^2}{4\Omega^2}K_u^2}+R_0^2\right)^{-1/2}$$

$$= \frac{2\Omega}{c}R_0^2\frac{\frac{c^2}{4\Omega^2}K_u^2}{1-\frac{c^2}{4\Omega^2}K_u^2}\left(\frac{R_0^2}{1-\frac{c^2}{4\Omega^2}K_u^2}\right)^{-3/2} - \frac{2\Omega}{c}\left(\frac{R_0^2}{1-\frac{c^2}{4\Omega^2}K_u^2}\right)^{-1/2}$$

$$= \frac{2\Omega}{c}\frac{c^2}{4\Omega^2}K_u^2\frac{R_0^2}{1-\frac{c^2}{4\Omega^2}K_u^2}\left(\frac{R_0^2}{1-\frac{c^2}{4\Omega^2}K_u^2}\right)^{-3/2} - \frac{2\Omega}{c}\left(\frac{R_0^2}{1-\frac{c^2}{4\Omega^2}K_u^2}\right)^{-1/2}$$

$$\phi''(u_0,\Omega) = \frac{2\Omega}{c}\frac{c^2}{4\Omega^2}K_u^2\left(\frac{R_0^2}{1-\frac{c^2}{4\Omega^2}K_u^2}\right)^{-1/2} - \frac{2\Omega}{c}\left(\frac{R_0^2}{1-\frac{c^2}{4\Omega^2}K_u^2}\right)^{-1/2}$$

$$= \frac{2\Omega}{c}\left(\frac{c^2}{4\Omega^2}K_u^2 - 1\right)\left(\frac{1-\frac{c^2}{4\Omega^2}K_u^2}{R_0^2}\right)^{-1/2}$$

$$= -\frac{2\Omega}{cR_0}\sqrt{1-\frac{c^2}{4\Omega^2}K_u^2}$$

so that finally

$$\phi''(u_0,\Omega) = -\frac{1}{R_0}\sqrt{\frac{4\Omega^2}{c^2} - K_u^2} \tag{21}$$

We can now apply Eqs. (20) and (21) in the PSP Eq. (12) to approximate the frequency response of Eq. (10):

$$H(K_u,\Omega;R_0) = \exp\left(+j\Omega\frac{2}{c}R_0\right)\int_{-\infty}^{\infty}\exp\left[j\left(-\Omega\frac{2}{c}\sqrt{u^2+R_0^2} - K_u u\right)\right]du$$

$$\cong \exp\left(+j\Omega\frac{2}{c}R_0\right)\sqrt{\frac{-2\pi}{\phi''(u_0,\Omega)}}\exp\left(-j\tfrac{\pi}{4}\right)\exp\left[j\phi(u_0,\Omega)\right]$$

$$= \exp\left(+j\Omega\frac{2}{c}R_0\right)\sqrt{\frac{2\pi}{\frac{1}{R_0}\sqrt{\frac{4\Omega^2}{c^2}-K_u^2}}}\exp\left(-j\tfrac{\pi}{4}\right)A(u_0)\exp\left(-jR_0\sqrt{\frac{4\Omega^2}{c^2}-K_u^2}\right) \tag{22}$$

$$= \sqrt{\frac{2\pi R_0}{\sqrt{\frac{4\Omega^2}{c^2}-K_u^2}}}\exp\left(-j\tfrac{\pi}{4}\right)(1)\exp\left[-jR_0\left(\frac{2\Omega}{c}+\sqrt{\frac{4\Omega^2}{c^2}-K_u^2}\right)\right]$$

The amplitude term can be further simplified by noting that $4\Omega^2/c >> K_u^2$. K_u^2 can thus be neglected in the amplitude term, reducing it to

$$\sqrt{\frac{2\pi R_0}{\sqrt{\frac{4\Omega^2}{c^2}-K_u^2}}} = \sqrt{\frac{2\pi R_0}{\frac{2\Omega}{c^2}\sqrt{1-\frac{K_u^2}{4\Omega^2/c^2}}}} = \sqrt{\frac{c\pi R_0}{\Omega}} \tag{23}$$

Combining Eqs. (22) and (23) and noting that $\exp\left(-j\tfrac{\pi}{4}\right) = \sqrt{1/j}$ gives the final form,

$$H\left(K_u, \Omega; R_0\right) = \sqrt{\frac{\pi c R_0}{j\Omega}} \exp\left\{ + jR_0 \left[\frac{2\Omega}{c} - \sqrt{\left(\frac{2\Omega}{c}\right)^2 - K_u^2} \right] \right\} \tag{24}$$

which is the desired result given in Eq. (8.50) of [1].

Reference [1] continues from this point to make further approximations to Eq. (24) to generate a separable form of the frequency response (Eq. (8.53) in [1]) that can be directly interpreted in terms of range migration correction and cross-range matched filtering, and that better illustrates the reason for the nomenclature "range-Doppler" for the algorithm. This approximation is adequately described in [1], so is not repeated here.

4 References

[1] M. A. Richards, *Fundamentals of Radar Signal Processing*. McGraw-Hill, New York, 2005.

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