Notes on Noncoherent Integration Gain

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July 2014

1 Noncoherent Integration Gain

Consider threshold detection of a radar target in additive complex Gaussian noise. To achieve a certain probability of detection and false alarm (P_D and P_{FA}, respectively) with a single measurement ($N = 1$), a certain signal-to-noise ratio (SNR_1) is required. The required value depends on the target radar cross section (RCS) statistics. Equations for P_D given P_{FA} and N and a square-law detector are given in [1] for a nonfluctuating target and for targets with RCS fluctuations modeled by one of the four Swerling models, also described in [1].

It is common in some radar systems to collect $N > 1$ measurements of the same range bin or range-Doppler bin and integrate (sum) them before performing the threshold test. If the phase of the data is discarded (i.e., the integration is performed after the square-law detector[1]), it is considered *noncoherent integration*. If the complex data is summed prior to the square-law detector, it is considered *coherent integration*. For either type of integration, the average single-measurement SNR required of each of the N measurements, SNR_N, is then less than SNR_1.[2]

Noncoherent integration gain,[3] denoted here as G_{nc}, refers to this reduction in the single-measurement average SNR required to meet a given P_D and P_{FA} when noncoherent integration is used. Specifically,

$$G_{nc} \equiv \frac{SNR_1}{SNR_N} \tag{1}$$

More information on G_{nc}, including equations for its estimation, is available in [2].

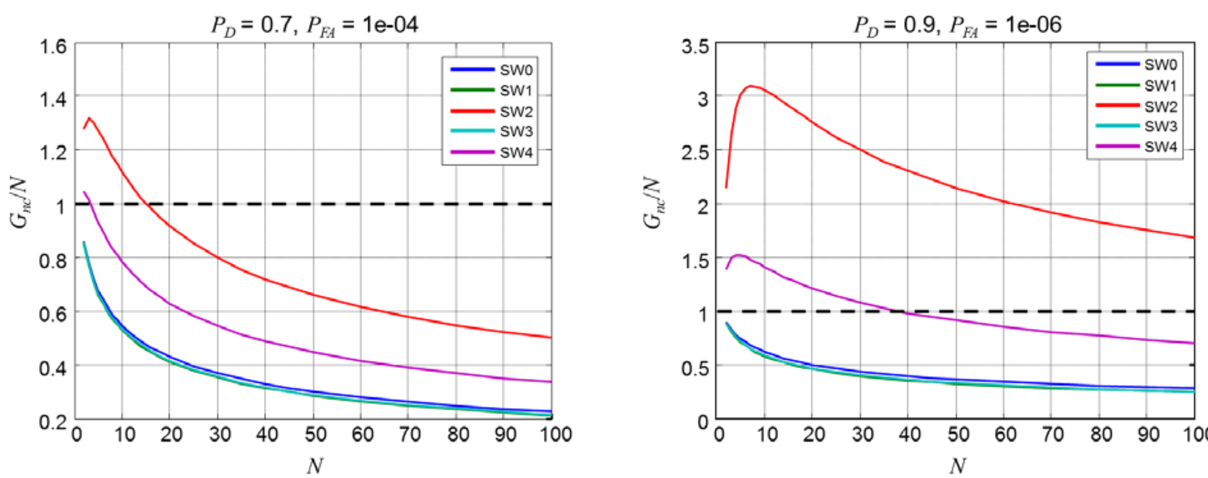
2 G_{nc} Can Be Greater Than N

It is often said[4] that G_{nc} is less than the factor of N observed with coherent integration of nonfluctuating targets, but greater than $\sqrt{N}$. It is well-known that the lower bound of $\sqrt{N}$ is too pessimistic, although G_{nc} may approach it asymptotically for very large N; see for instance [1] and [3]. The principal point of this memo is that the upper limit of N holds for nonfluctuating targets and for Swerling 1 and 3 targets, but does *not* hold for Swerling 2 and 4 targets; in these latter cases, G_{nc} is greater than N, sometimes

[1] Linear and log detectors are also common. We restrict our attention to the more mathematically-tractable square-law detector here. Similar results can be expected to hold for a linear detector. The author has not investigated the log detector case.
[2] Note that both SNR_1 and SNR_N are the SNR of a single measurement. The subscript refers to the number of such measurements integrated for a single threshold test.
[3] G_{nc} is called the *integration improvement factor* (IIF) in some references. The term IIF may also be applied to coherent integration, and so is a more general term.
[4] Including by this author; see [1], pages 324-326, and [2].

significantly greater, for certain ranges of P_D and N for a given P_{FA} [3]. The Swerling 1 and 3 cases are those in which the target RCS, while drawn from either the exponential (Swerling 1) or 4th-degree chi-square (Swerling 3) probability density function (PDF), has a single value for all N measurements that are integrated for one threshold test. These models exhibit noncoherent integration gain similar to that of the nonfluctuating case, in which the RCS is a single fixed value (not a random value drawn from a PDF) for all N measurements integrated. The nonfluctuating case is often referred to as the Swerling 0 of Swerling 5 case. In contrast to these fully correlated models, the Swerling 2 and 4 cases are models where each of the measurement RCS values is an *independent* random variable drawn from either the exponential (Swerling 2) or 4th-degree chi-square (Swerling 4) PDF. Multiple different random variables are summed in these cases and the noncoherent integration gain behavior is significantly different.

Figure 1 illustrates these statements by plotting the normalized noncoherent integration gain G_{nc}/N vs. N for two different choices of P_D and P_{FA} and for all four Swerling models of data fluctuation as well as the nonfluctuating model.[5] Values greater than one therefore represent noncoherent integration gains greater than a factor of N. In this and subsequent figures, "SW0" refers to the Swerling 0 case, "SW1" to Swerling 1, and so forth. This figure shows that gains greater than a factor of N are obtained only in the Swerling 2 and 4 cases, and then only for N less than some value that is strongly dependent on P_{FA}, P_D, and whether the measurement PDF is exponential (Swerling 2) or 4th-degree chi-square (Swerling 4). In the Swerling 1 and 3 cases, it remains true that integration gain is less than a factor of N. This is also true of the nonfluctuating case. Consequently, attention is focused mostly on the Swerling 2 and 4 cases in the remainder of this memo.

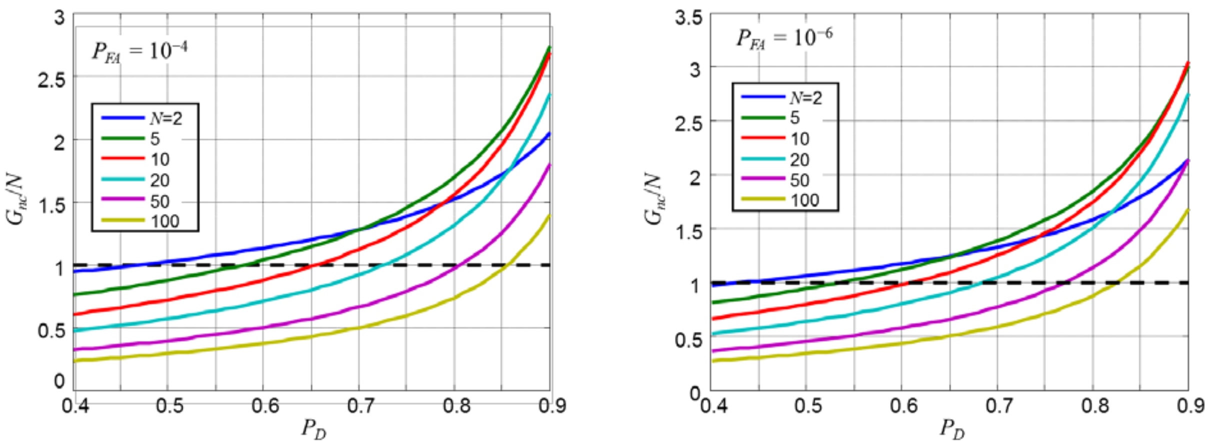


(a) (b)

Figure 1. Normalized noncoherent integration gain G_{nc}/N vs. N for a square-law detector, two values of P_D and P_{FA}, and five different measurement fluctuation models. (a) P_D = 0.7 and $P_{FA} = 10^{-4}$. (b) P_D = 0.9 and $P_{FA} = 10^{-6}$.

[5] The Swerling 1 and 3 cases, while not identical, are so closely similar that their lines are indistinguishable on these plots.

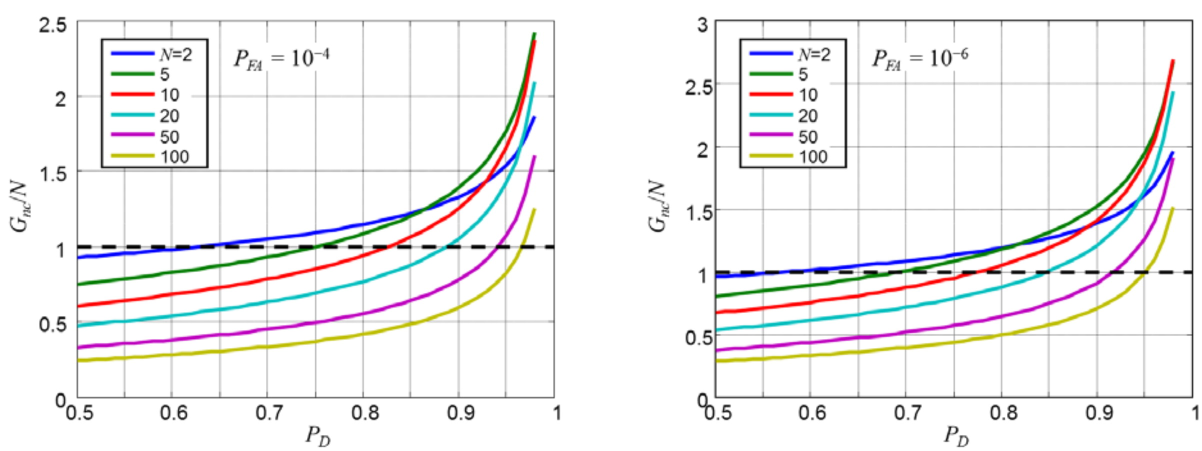
Figure 2(a) shows the variation of the normalized gain G_{nc}/N with P_D for a Swerling 2 target, square-law detector, $P_{FA} = 10^{-4}$, and various values of N. This figure shows that for $P_{FA} = 10^{-4}$ and P_D values ranging from about 47% for $N = 2$ to about 86% for $N = 100$, a Swerling 2 target can achieve integration gains greater than N. For smaller values of P_{FA} the trends are very similar, although the integration gains for high P_D are even larger and the minimum values of P_D for which the gain exceeds N are slightly lower. This is seen in Fig. 2(b), where $P_{FA} = 10^{-6}$.



(a) (b)

Figure 2. Variation of G_{nc}/N with P_D for a Swerling 2 target, square-law detector, and various values of N. (a) $P_{FA} = 10^{-4}$. (b) $P_{FA} = 10^{-6}$.

Figure 3 shows similar results for Swerling 4 fluctuations. In the Swerling 4 case the noncoherent integration gains are not as large and the P_D required before a gain greater than N is achieved is larger.



(a) (b)

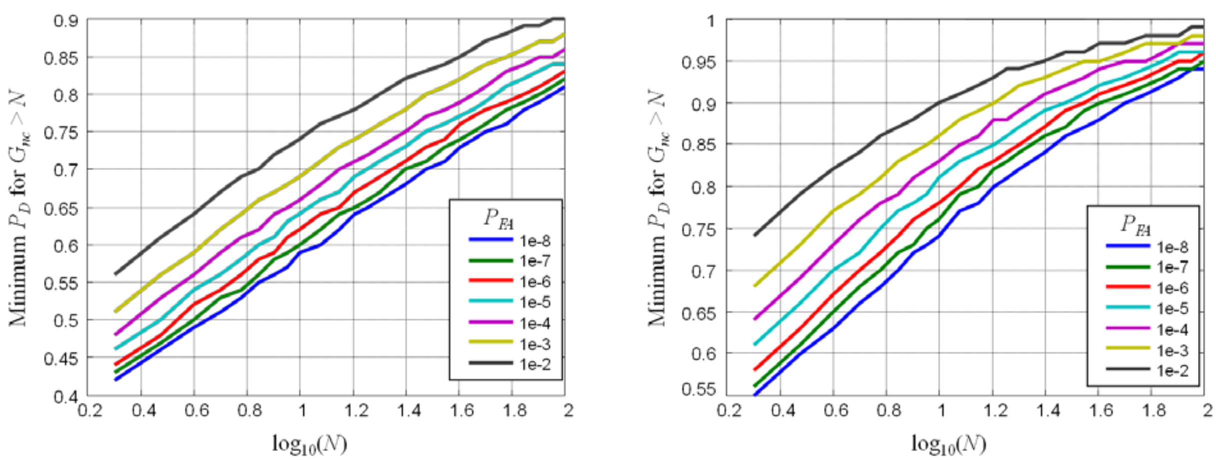
Figure 3. Variation of G_{nc}/N with P_D for a Swerling 4 target, square-law detector, and various values of N. (a) $P_{FA} = 10^{-4}$. (b) $P_{FA} = 10^{-6}$.

3 Values of P_D or P_{FA} for which $G_{nc} > N$

Figure 4(a) plots $P_{D\min}$, the minimum value of P_D for which $G_{nc} > N$ for a Swerling 2 target, as a function of the base 10 logarithm of N for a fixed value of P_{FA}.[6] The range of $\log_{10}(N)$ shown corresponds to N ranging from 2 to 100. For a given choice of N and P_{FA}, $G_{nc} > N$ for all values of P_D greater than $P_{D\min}$. For example, with $N = 10$ and $P_{FA} = 10^{-3}$, $G_{nc} > 10$ for $P_D \geq 0.7$. Clearly $P_{D\min}$ is approximately linear in $\log_{10}(N)$. Although not shown here, examination of a similar plot with $\log_{10}(P_{FA})$ as the abscissa shows that it is also approximately linear in $\log_{10}(P_{FA})$ for smaller values of P_{FA}. The increased spacing between lines at larger false alarm rates shows that this linearity breaks down at higher values of P_{FA}. Nonetheless, it is somewhat useful to model $P_{D\min}$ as a linear function of both $\log_{10}(P_{FA})$ and $\log_{10}(N)$. This is easily done using the MATLAB® curve fitting tool `cftool`. The result is

$$P_{D\min} = 0.53 + 0.022\log_{10}\left(P_{FA}\right) + 0.222\log_{10}\left(N\right) \tag{2}$$

Although entirely empirical and of limited accuracy, Eq. (2) is nonetheless useful in estimating roughly the combinations of P_D, P_{FA} and N for which noncoherent integration gains greater than N can be expected for Swerling 2 data fluctuations. Table 1 gives sample estimates of $P_{D\min}$ computed with Eq. (2). Comparison with the appropriate curves in Fig. 2 suggests that Eq. (2) can estimate $P_{D\min}$ to within 3 or 4%, at least in this range of P_{FA} and N.



(a) (b)

Figure 4. Minimum detection probability to achieve a noncoherent integration gain greater than *N* for Swerling 2 fluctuations.

[6] These curves and others in this memo were computed using the functions in the "Detection Calculator" section of the free MATLAB® supplements software available at www.radarsp.com.

Table 1. Estimated Minimum P_D to Achieve $G_{nc} > N$ for Swerling 2 Fluctuations

N	2		10		100	
P_{FA}	10^{-4}	10^{-6}	10^{-4}	10^{-6}	10^{-4}	10^{-6}
$P_{D\min}$ (estimated)	0.51	0.46	0.66	0.62	0.89	0.84

Figure 4(b) presents a similar plot for the Swerling 4 case. The variation with $\log_{10}(N)$ is not as linear for large N due to the generally higher P_D required to achieve $G_{nc} > N$ for a given P_{FA} and large N, and the resulting compression of the curves as the minimum P_D approaches 1. Although not illustrated, the linearity in $\log_{10}(P_{FA})$ is similar to that of the Swerling 2 case. A bilinear fit to the Swerling 4 data gives the estimate for $P_{D\min}$ of Eq. (3). Table 2 gives sample estimates using this equation, which can be compared to the data of Fig. 2. For the examples shown, the estimate error is no more than approximately 3 to 5% over much of the parameter range, but increases to as much as 7% for very small N and the larger values of P_{FA}.

$$P_{D\min} = 0.71 + 0.02\log_{10}\left(P_{FA}\right) + 0.195\log_{10}\left(N\right) \tag{3}$$

Table 2. Estimated Minimum P_D to Achieve $G_{nc} > N$ for Swerling 4 Fluctuations

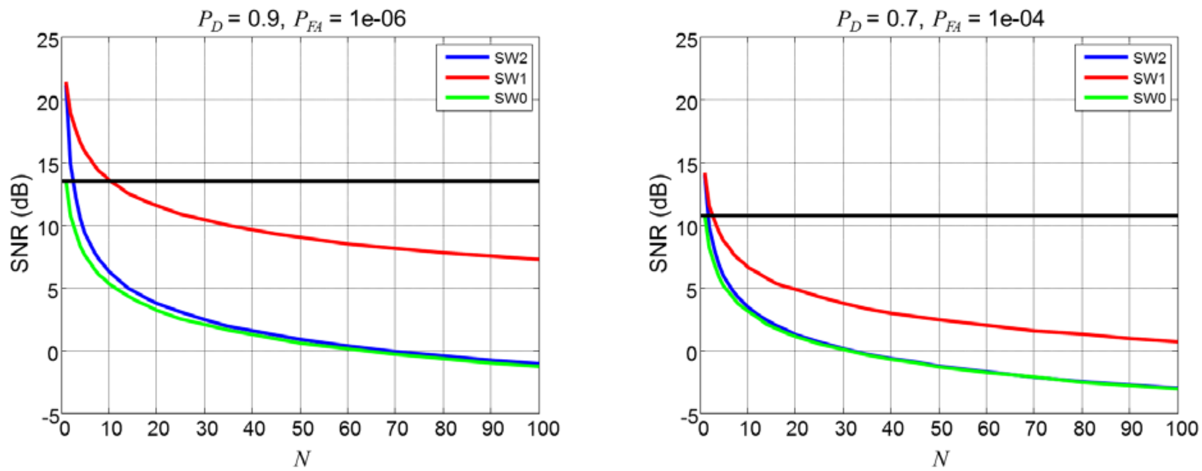
N	2		10		100	
P_{FA}	10^{-4}	10^{-6}	10^{-4}	10^{-6}	10^{-4}	10^{-6}
$P_{D\min}$ (estimated)	0.69	0.65	0.83	0.79	1.0	0.98

4 No Free Lunch, Part 1: High Integration Gain Does Not Mean Low SNR

The fact that G_{nc} can be greater than N for some ranges of P_D and N for a given P_{FA} does not imply that the measurement-to-measurement fluctuations of the Swerling 2 and Swerling 4 cases result in better detection performance than either noncoherent integration of a nonfluctuating target, or coherent integration of a nonfluctuating target. While the value of SNR required may improve more rapidly than a factor of N (see Figure 1), it remains higher than in the nonfluctuating cases, even if only slightly so.

Figure 5 illustrates the required average value of single-sample SNR, SNR_N, needed to achieve the indicated P_D and P_{FA} vs. the number N of measurements noncoherently integrated for a square-law detector and three different Swerling models of target RCS behavior, the nonfluctuating case and the two exponential-PDF models. The nonfluctuating case achieves the desired detection performance with the lowest value of SNR_N for any value of N. For $N = 1$, the value of SNR_N required for the Swerling 1 and Swerling 2 cases is approximately 8 dB above that required for the nonfluctuating case for $P_D = 0.9$ and $P_{FA} = 10^{-6}$. For the Swerling 2 case, this difference shrinks rapidly, to about 4 dB at $N = 2$, 3 dB at $N = 3$, and 1 dB at $N = 10$, but the required SNR_N is always greater than for the nonfluctuating case. For the Swerling 1 case, the SNR difference remains high and even increases slightly with N, reaching about 8.5

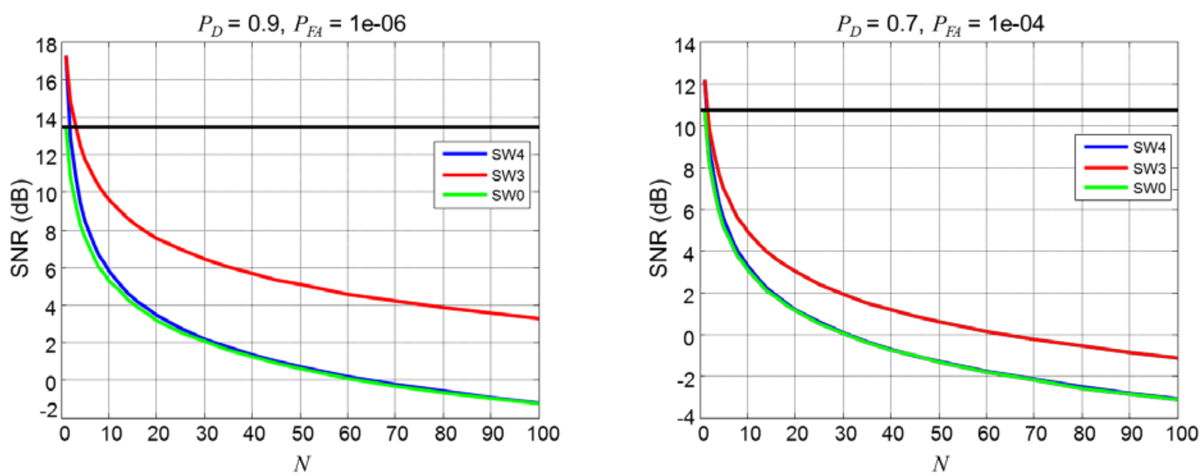
dB at $N = 100$. The maximum difference is smaller, in the vicinity of 3 dB or so at $N = 1$, when $P_D = 0.7$ and $P_{FA} = 10^{-4}$.



(a) (b)

Figure 5. Average single-measurement SNR required to achieve a specified value of P_D and P_{FA} vs. number N of measurements noncoherently integrated for a square-law detector and three different Swerling models of target RCS behavior. (a) $P_D = 0.9$ and $P_{FA} = 10^{-6}$ (b) $P_D = 0.7$ and $P_{FA} = 10^{-4}$

Figure 6 shows that similar behavior is observed for Swerling 3 and 4 fluctuations, which use a 4th-degree chi-square PDF. The maximum difference between the nonfluctuating case and the Swerling 3 case is smaller, around 4 dB for $P_D = 0.9$ and $P_{FA} = 10^{-6}$ and 2 dB for $P_D = 0.7$ and $P_{FA} = 10^{-4}$.



(a) (b)

Figure 6. Average single-measurement SNR required to achieve a specified value of P_D and P_{FA} vs. number N of measurements noncoherently integrated for a square-law detector and three different Swerling models of target RCS behavior. (a) $P_D = 0.9$ and $P_{FA} = 10^{-6}$ (b) $P_D = 0.7$ and $P_{FA} = 10^{-4}$

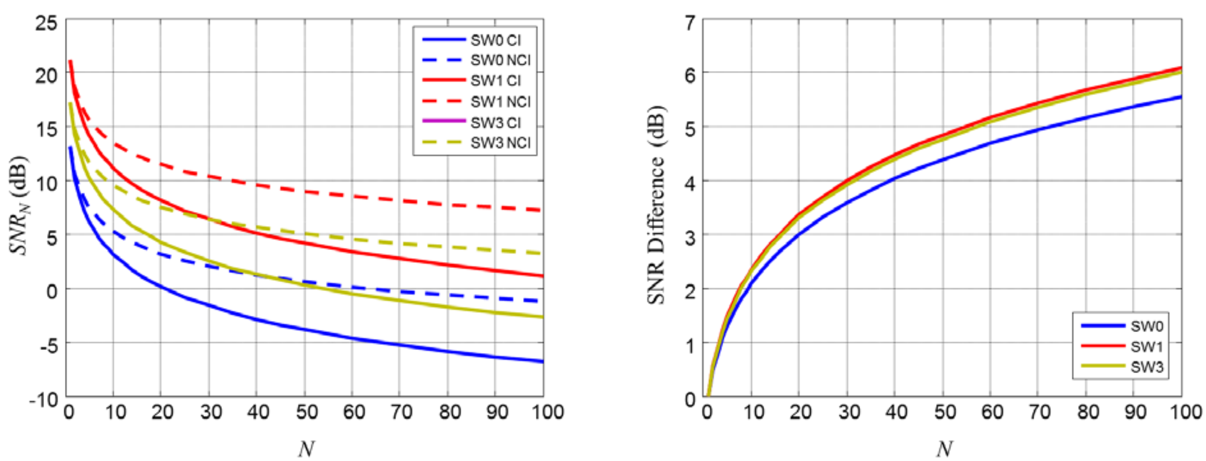
5 No Free Lunch, Part 2: Coherent Integration is Still Better

Coherent integration assumes that the data acquisition protocol is such that target measurements can be added in phase. Successful coherent integration of a nonfluctuating target achieves an integration gain of exactly a factor N. Given the larger noncoherent integration gains obtained in some situations, it is of interest to compare the SNR_N required for a nonfluctuating target using coherent integration to that required of a fluctuating target using noncoherent integration.

As usually used, the concept of coherent integration implicitly assumes a nonfluctuating target. However, it can also be applied to fluctuating targets in the Swerling 1 and 3 cases, which assume the target RCS does not fluctuate within a group of N measurements combined for one threshold detection test. The single-measurement SNR required for coherent integration of N samples for a given P_D, P_{FA}, and N can be determined as follows:

1. Determine the SNR required to achieve the desired P_D and P_{FA} using $N = 1$; call this SNR_{coh}.
2. Then $SNR_N = SNR_{\text{coh}}/N$.

Figure 7 shows that the single-measurement SNR required using coherent integration is less than the average single-measurement SNR required using noncoherent integration for all three relevant Swerling models. Part (a) of the figure shows the required SNR_N. In this plot, paired solid and dashed lines of the same color represent the same Swerling model with coherent and noncoherent integration, respectively. Part (b) of the figure simply plots the difference in decibels between each pair of curves; this represents the extra SNR required using noncoherent integration. These plots show that the single-pulse SNR required is always less with coherent integration than with noncoherent integration.



(a) (b)

Figure 7. Comparing single-measurement SNR required for coherent and noncoherent integration with $P_D = 0.9$ and $P_{FA} = 10^{-6}$ vs. number N of measurements noncoherently integrated for a square-law detector and three different Swerling models of target RCS behavior. (a) SNR_1 for both types of integration and Swerling 0, 1, and 3 models. (b) Increase in SNR_1 required using noncoherent integration over that required using coherent integration.

It is no surprise that coherent integration is always better than noncoherent with the Swerling 0, 1, and 3 cases, in which the N measurements to be integrated have the same value. Noncoherent gain factors greater than N were observed only for the Swerling 2 and 4 cases. One might then ask whether noncoherent integration gain can exceed coherent integration gain in the Swerling 2 and 4 cases. At least in the Swerling 2 case, the answer is "no". In that case, the pre-detector measurements are zero-mean complex Gaussian random variables. Integrating them coherently (i.e., summing these complex values) simply produces another zero-mean complex Gaussian random variable, with a variance that is N times that of the individual measurements. Complex noise has exactly the same form and produces the same rate of power growth. Therefore, the SNR is unchanged.

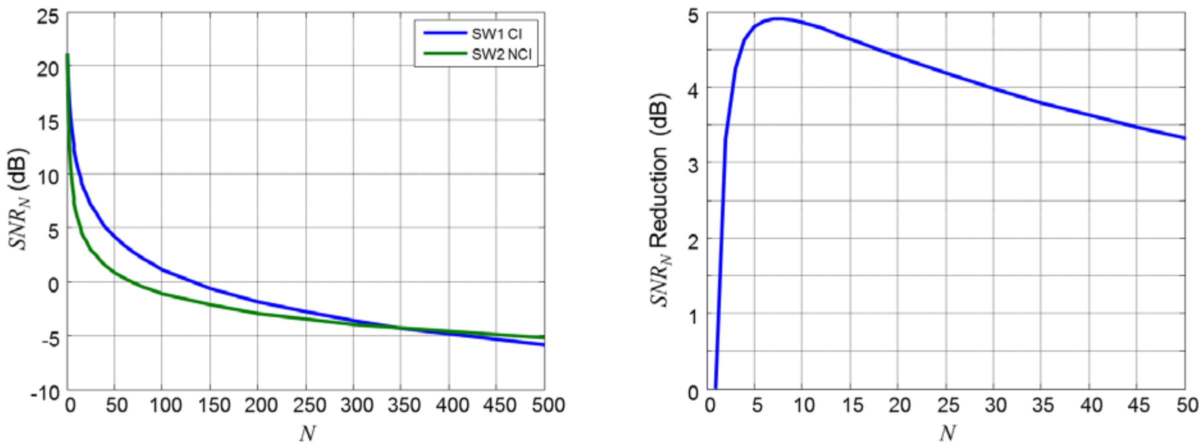
For the Swerling 4 case, the pre-detector statistics of the complex data are less clear. One can create simulated data having a uniform random phase and an amplitude that is the square root of a 4^{th}-degree chi-square random variable. Such data also exhibits the factor-of-N variance growth when N samples are integrated, so that there would again be no coherent integration gain vs. noise. However, it is not clear the uniform phase is a reasonable assumption. The Swerling 4 model is motivated by the idea of a dominant scatter modulated by a number of smaller scatterers, suggesting the phase might have only a limited spread around the value of the dominant scatterer phase. It is therefore possible that a more reasonable phase assumption would be some member of the Tikhonov family, and that the results of [4] may be useful in determining the achievable integration gain.

6 To Frequency Hop, or Not to Frequency Hop?

It is well known that frequency agility can be used to cause a target exhibiting Swerling 1 or 3 measurement fluctuations (constant within a group of N measurements) to instead exhibit Swerling 2 or 4 measurement-to-measurement fluctuations, respectively [1],[3]. While the coherent integration gain factor of N is more efficient than noncoherent integration for a Swerling 1 or 3 target (see Figure 1 or Figure 7), it is also true that noncoherent integration of Swerling 2 or 4 data achieves a noncoherent integration gain factor greater than N (Figure 1 through Figure 4). One can then ask whether, for a Swerling 1 target, it is more effective to apply coherent integration or to use frequency agility to convert it to Swerling 2 behavior and then use noncoherent integration? The same question arises for Swerling 3/4 targets.

The answer depends, for a given P_D and P_{FA}, on the value of N. Figure 8 compares the value of SNR_N needed using coherent integration for Swerling 1 data vs. that needed for noncoherent integration of Swerling 2 data when $P_D = 0.9$ and $P_{FA} = 10^{-6}$. Part (a) of the figure shows that the average single-measurement SNR needed is less for Swerling 2 if N is less than approximately 360. Thus, if fewer than 360 samples are to be collected and integrated, it would be preferable from the point of view of SNR requirements to use frequency agility to decorrelate the measurements and then integrate them noncoherently.

If more than 360 samples are available it is preferable to use a constant radar frequency so the measurements remain correlated and then integrate them coherently. This strategy assumes that the data will remain coherent over the possibly lengthy time period required to obtain several hundred measurements; if this is not the case, frequency agility and noncoherent integration is likely preferred for any N.



(a) (b)

Figure 8. Comparing single-measurement SNR required for coherent integration of a Swerling 1 target and noncoherent integration of a Swerling 2 target, both with P_D = 0.9 and $P_{FA} = 10^{-6}$ and a square-law detector. (a)SNR_1 vs. N. (b) Reduction in required SNR_1 for noncoherent integration vs. coherent integration.

Other costs of using noncoherent integration instead of coherent integration must also be kept in mind. This is primarily the inability to compute a Doppler spectrum, and therefore to detect multiple targets in the same range bin, estimate radial velocities, and separate targets from clutter.

For less demanding detection requirements, e.g. $P_D = 0.7$ and $P_{FA} = 10^{-4}$, the qualitative behavior is similar but the crossover between constant frequency/coherent integration and frequency agility/noncoherent integration occurs at the much lower value of $N = 15$. In this case, coherent integration might be much more practical due to the shorter time involved before it becomes preferable to frequency agility with noncoherent integration.

A comparison of coherent integration of Swerling 3 measurements to noncoherent integration of Swerling 4 measurements produces all of the same qualitative behaviors. The crossover values of N are much lower in this case, and the difference in SNR_N values is also significantly less.

Returning to the Swerling 1/Swerling 2 comparison at $P_D = 0.9$ and $P_{FA} = 10^{-6}$, Fig. 8(b) shows the reduction in SNR_N possible with frequency agility and noncoherent integration at lower values of N. This curve has a broad maximum between 4.8 to 4.9 dB for N in the range 5 to 11, with a maximum of 4.91 for N equal to both 7 and 8.

Based on these observations, a hybrid strategy that combines a coherent spectrum with rapid noncoherent integration gain was suggested by G. A. Showman. Suppose the radar/target encounter is such that one could collect 480 measurements from a target whose RCS is characterized by an exponential PDF. Assume the parameters are such that the maximum SNR advantage for noncoherent integration occurs for $N = 8$ as in the example above. Then instead of collecting that data as one large 480-measurement coherent processing interval (CPI), instead divide it into eight 60-measurement CPIs and collect each at a different RF frequency. Form the 60-point (or larger, if zero padding is used) discrete Fourier transform (DFT) in each range bin of each CPI to obtain the range-Doppler spectrum for that CPI. An eight-fold noncoherent integration across CPIs is then performed in each range-Doppler bin. Interpolation in Doppler or zero padding will be necessary since the radar frequency change will change the Doppler scale form one CPI to the next.

Figure 9 plots the single-sample average SNR required to achieve $P_D = 0.9$ and $P_{FA} = 10^{-6}$ for all such possible decompositions of a set of 480 measurements. Each bar is labeled with a pair of numbers, e.g. (8,60) in the seventh bar from the left. The first number is the number of CPIs formed and is therefore also the degree of noncoherent integration. The second number is the size of the resulting CPIs, in this case 60 measurements per CPI for a total of 8×60 = 480 measurements. The (8,60) case is in fact the one allowing the lowest value of SNR_N, namely −10.57 dB. The minimum is broad, with little variation in the required SNR for noncoherent integration of between 4 and 20 CPIs.

The value of −10.57 dB arises as follows. The SNR required to achieve $P_D = 0.9$ and $P_{FA} = 10^{-6}$ for a Swerling 1 target with a single measurement is 21.15 dB. The SNR required when integrating 8 measurements of a Swerling 2 target (because we assume frequency agility is used to decorrelate these measurements) is 7.21 dB, representing a noncoherent integration gain of 13.94 dB (a factor of 24.77, or about 3.1× the value of N, consistent with Fig. 1(b)). The coherent integration gain for each 60-measurement CPI is $10\log_{10}(60) = 17.78$ dB. Therefore, the required SNR is

$$SNR = 21.15 - 13.94 - 17.78 = -10.57 \text{ dB}, \tag{4}$$

in agreement with Figure 9.

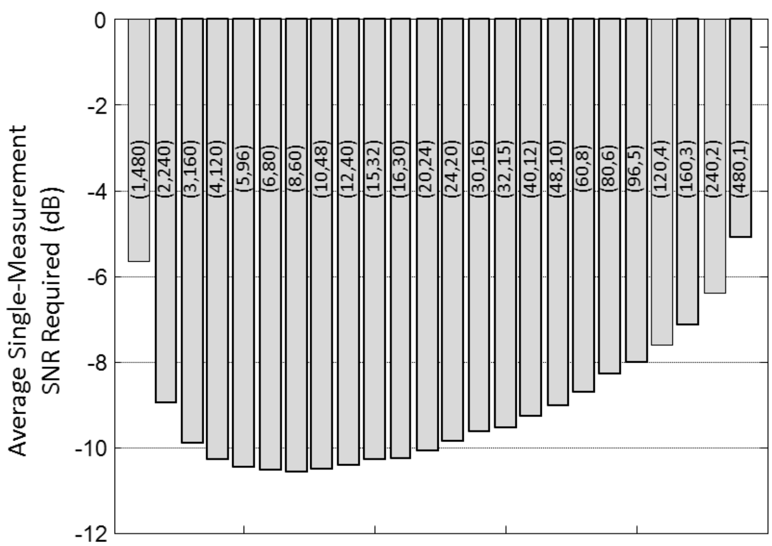


Figure 9. Average single-measurement SNR required for combined coherent and noncoherent integration of 480 measurements for a frequency-agile system, Swerling 1/2 target, P_D = 0.9 and P_{FA} = 10^{6} and a square-law detector. See text for discussion.

This hybrid approach combines the additional integration gain of Swerling 2 noncoherent integration for small N with the ability to form a Doppler spectrum and therefore to detect multiple targets and estimate radial velocity. The Doppler resolution will not be as fine as would be obtained with a DFT of the entire 480 measurements, but the single-measurement average SNR required for a given P_D and P_{FA} will be less, and the coherence requirements on the data will be reduced due to the shorter coherent integration time.

The optimum amount of noncoherent integration varies somewhat with P_D for a given P_{FA}. Lower values of P_D lead to lower optimum degrees of noncoherent integration. For $P_{FA} = 10^{-6}$ and 480 measurements, for example, the optimum degree of noncoherent integration ranges from 2 at $P_D = 0.5$ to 15 at $P_D = 0.99$. On the other hand, the optimum degree of noncoherent integration appears largely insensitive to both the total number of measurements and the P_{FA}.

7 Acknowledgement

The author is grateful (as usual) to Dr. Gregory A. Showman of the Georgia Tech Research Institute for many discussions about, and much insight into, the issue of noncoherent integration gain.

8 References

[1] M. A. Richards, *Fundamentals of Radar Signal Processing*, second edition. McGraw-Hill, 2014.

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