Relationship Between the Gamma, Erlang, Chi-Square, and Swerling 3/4 Probability Density Functions

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1 Background

The Swerling 3 and 4 models for radar cross section (RCS) fluctuation describe the RCS σ with a probability density function (pdf) that, in the radar literature, is commonly called a "4th degree chi-squared" pdf [1]. This terminology is a bit at odds with more common use of the name "chi-squared" for a pdf. The following sections explain the link between the Swerling 3/4 pdf and common terminology for the gamma, Erlang, and chi-squared pdf's.

While we are at it, we will show the relationship between the Swerling 3/4 pdf and the Rice pdf, which more accurately models the "dominant-plus-many small scatterers" physical model for which these Swerling models are intended.

2 The Gamma PDF

The gamma pdf $\Gamma(\alpha,\beta)$ for a random variable x is [2]

$$p_x(x) = \frac{x^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)}\exp(-x/\beta) \tag{1}$$

where $\Gamma(\cdot)$ is the usual gamma function [2]. The mean and variance of x are

$$\bar{x} = \alpha\beta, \quad \sigma_x^2 = \alpha\beta^2 \tag{2}$$

The gamma pdf arises in processes for which the waiting times between Poisson distributed events are relevant. In radar it is a generalization of common pdf's for describing RCS, such as the Erlang and the Swerling 3/4 pdf, described below.

3 The Erlang PDF

The Erlang or Erlang-k pdf $E(k,\lambda)$ for a random variable x is [2]

$$p_x(x) = \frac{(\lambda k)^k\, x^{k-1}}{\Gamma(k)}\exp(-\lambda k x) \tag{3}$$

Recall that for integer k, $\Gamma(k) = (k-1)!$. The mean and variance of x are

$$\bar{x} = \frac{1}{\lambda}, \quad \sigma_x^2 = \frac{1}{k\lambda^2} \tag{4}$$

The Erlang pdf arises is also related to waiting times for Poisson-distributed events. In radar, it is the pdf for the noncoherent sum of nonfluctuating target echoes in noise.

The Erlang pdf is a special case of the gamma pdf. Specifically,

$$E(k,\lambda) = \Gamma\left(k, \frac{1}{\lambda k}\right) \tag{5}$$

4 The N^{th}-Degree Chi-Squared PDF

The common definition of the chi-squared pdf with N degrees of freedom $\Xi(N)$ (also called the chi-squared of duo-degree N) is [2]

$$p_x(x) = \frac{x^{N/2-1}}{2^{N/2}\,\Gamma(N/2)} \exp(-x/2) \tag{6}$$

Recall that for integer k, $\Gamma(k) = (k-1)!$. The mean and variance of x are

$$\bar{x} = N, \quad \sigma_x^2 = 2N \tag{7}$$

The N^{th}-degree chi-square pdf describes the sum of the squares of N zero-mean, unit-variance Gaussian random variables.

The N^{th}-degree chi-square is a special case of the Erlang pdf, and thus also of the gamma pdf. Specifically,

$$\Xi(N) = E\left(\frac{N}{2}, \frac{1}{N}\right) = \Gamma\left(\frac{N}{2}, 2\right) \tag{8}$$

The 4^{th}-degree chi-squared is of special interest. Setting $N = 4$ gives

$$p_x(x) = \frac{x}{4} \exp(-x/2) \tag{9}$$

$$\bar{x} = 4, \quad \sigma_x^2 = 8 \tag{10}$$

$$\Xi(4) = E\left(2, \frac{1}{4}\right) = \Gamma(2,2) \tag{11}$$

5 The Swerling 3/4 PDF

The Swerling 3/4 pdf for RCS σ is

$$p_\sigma(\sigma) = \frac{4\sigma}{\bar{\sigma}^2}\exp\left(-2\sigma/\bar{\sigma}\right) \tag{12}$$

where $\bar{\sigma}$ is the mean RCS and the variance is $\bar{\sigma}^2/2$. Calling this a 4th-degree chi-square pdf is common in radar but is otherwise somewhat non-standard terminology. A chi-square of degree N is usually considered to be a special case of the gamma pdf $\Gamma(\alpha,\beta)$ with $\alpha = N/2$ and $\beta = 2$ as described above. The Swerling 3/4 pdf is $\Gamma\left(2,\bar{\sigma}/2\right)$, which is 4th degree but does not have $\beta = 2$. The more general form of the so-called "shape parameter" β is necessary to allow the mean of the distribution to be set to any desired value.

The Swerling 3/4 pdf is also a special case of the Erlang-k distribution, specifically $E(2, 1/\bar{\sigma})$.

6 Relationship of the Swerling 3/4 PDF to the Rice PDF

The Swerling 3/4 pdf is intended to model the observed RCS fluctuations of a target with a single dominant scatterer and many contributing smaller scatterers. Thus, the composite echo is expected to be the sum of a large constant with many smaller terms, and with all of the relative phases varying uniformly over the interval $[0,2\pi)$. The correct pdf for this situation is the Rice or Rician pdf. In one form, this is [1]

$$p_\sigma(\sigma) = \frac{1}{\bar{\sigma}}\left(1+a^2\right)\exp\left[-a^2\frac{\sigma}{\bar{\sigma}}\left(1+a^2\right)\right]I_0\left[2a\sqrt{\left(1+a^2\right)\left(\sigma/\bar{\sigma}\right)}\right] \tag{13}$$

The parameter a is the ratio of the total power in the dominant scatterer to that of all of the smaller scatterers combined. This pdf is also common in modeling fading in communication channels. The mean is $\bar{\sigma}$ and the variance is

$$\operatorname{var}(x) = \frac{\left(1+2a^2\right)}{\left(1+a^2\right)^2}\bar{\sigma}^2 \tag{14}$$

The presence of a Bessel function in the pdf complicates its use for deriving detection curves. Swerling sought a simpler pdf that would be similar to the Rician. He chose the chi-squared with 4 degrees of freedom, generalized as above to allow for an arbitrary mean. The first two moments (mean and variance) of the two distributions match if the ratio a of the dominant to small scatterers in the Rician model is $1+\sqrt{2}$, as can be readily verified by equating their variances when they have the same mean $\bar{\sigma}$. Thus, the 4th-degree generalized chi-square used by Swerling best matches the Rician only for a specific value of the dominant scatterer RCS to the small-scatterer RCS.

7 References

[1] M. A. Richards, *Fundamentals of Radar Signal Processing*. McGraw-Hill, New York, 2005.

[2] A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes*. McGraw-Hill, New York, 2001.