Exact and Approximate Detection Probability Formulas in *Fundamentals of Radar Signal Processing*

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1 Introduction

Table 6.1 in the text *Fundamentals of Radar Signal Processing*, 2nd ed. [1], is reproduced below. It gives formulas for the probability of false alarm P_{FA} and the probability of detection P_D for the four standard Swerling target fluctuation models and the nonfluctuating model (denoted Swerling 0 or 5, as is often don) in additive Gaussian noise when a square law detector and N-fold noncoherent integration are used. $\bar{\chi}$ is the signal-to-noise ratio. The same table also appears as Table 6.1 of the first edition of the book.

Table 6.1 from [1], "Probability of Detection for Swerling Model Fluctuating Targets with a Square-Law Detector".

Case	P_D	Comments
0 or 5	$Q_M\left(\sqrt{2N\bar{\chi}},\sqrt{2T}\right)+e^{-(T+N\bar{\chi})}\sum_{r=2}^{N}\left(\frac{T}{N\bar{\chi}}\right)^{\frac{r-1}{2}} I_{r-1}\left(2\sqrt{NT\bar{\chi}}\right)$	Nonfluctuating case
1	$\left(1+\frac{1}{N\bar{\chi}}\right)^{N-1}\exp\left[\frac{-T}{1+N\bar{\chi}}\right]$	Approximate for $P_{FA} \ll 1$ and $N\bar{\chi} > 1$; exact for $N = 1$
2	$1-I\left[\frac{T}{(1+\bar{\chi})\sqrt{N}},N-1\right]$	
3	$\left(1+\frac{2}{N\bar{\chi}}\right)^{N-2}\left[1+\frac{T}{1+(N\bar{\chi}/2)}-\frac{2(N-2)}{N\bar{\chi}}\right]\exp\left[\frac{-T}{1+N\bar{\chi}/2}\right]$	Approximate for $P_{FA} \ll 1$ and $N\bar{\chi}/2 > 1$; exact for $N = 1$ or 2
4	$c^N\sum_{k=0}^{N}\frac{N!}{k!(N-k)!}\left(\frac{1-c}{c}\right)^{N-k}\left\{\sum_{l=0}^{2N-1-k}\frac{e^{-cT}(cT)^l}{l!}\right\}\quad T > N(2-c)$ $1-c^N\sum_{k=0}^{N}\frac{N!}{k!(N-k)!}\left(\frac{1-c}{c}\right)^{N-k}\sum_{l=2N-k}^{\infty}\frac{e^{-cT}(cT)^l}{l!}\quad T < N(2-c)$	$c \equiv \frac{1}{1+(\bar{\chi}/2)}$
$P_{FA} = 1-I\left(\frac{T}{\sqrt{N}},N-1\right)$ in all cases		
$I(\cdot,\cdot)$ is Pearson's form of the incomplete Gamma function; $I_k(\cdot)$ is the modified Bessel function of the first kind and order k.		

The particular forms of the detection equations in this table were obtained from the text by Meyer and Mayer ("M&M") [2]. However, the versions presented in this table, while useful, are a somewhat inconsistent mix of exact and approximate results. The first goal of this memo is to present a more consistent set of exact results, and to show briefly how some approximate, simplified results are then derived.

A second issue is that [2] and therefore the original table uses Pearson's form of the incomplete gamma function to express some of the results. Pearson's incomplete gamma function is defined in [2] as

$$I_P(a,b) = \frac{1}{b!}\int_0^{a\sqrt{b+1}} t^b e^{-t} dt = \frac{1}{\Gamma(b+1)}\int_0^{a\sqrt{b+1}} t^b e^{-t} dt \tag{1}$$

Meyer and Mayer use only the first version because the argument b is always integer; the second version is a generalization for non-integer b. Despite the $I_P(\cdot,\cdot)$ notation adopted here, the table above uses just $I(\cdot,\cdot)$ for Pearson's form because that is the way it was presented in [1].

It now is more common to use an incomplete gamma function or "normalized" incomplete gamma function defined as [3][4]

$$I(c,d) = \frac{1}{\Gamma(d)}\int_0^{c} t^{d-1} e^{-t} dt = \frac{1}{(d-1)!}\int_0^{c} t^{d-1} e^{-t} dt \tag{2}$$

where the second version is for an integer argument d, which will always be our case. Eqn. (2) is also the definition used by MATLAB™ for their `gammainc` function. The two versions are related by

$$I_P(a,b) = I\left(a\sqrt{b+1}, b+1\right) \tag{3}$$

This conversion is also noted in [5]. The second goal of this memo is to restate the tabular results in terms of the more modern, common, and MATLAB™-compatible normalized incomplete gamma function.

2 Exact Equations for P_D, in Original Pearson's and "Modern" MATLAB™-Compatible Forms

The formulas for P_D in the Swerling 1 and 3 cases of the original Table 6.1 above are approximations, as noted in the last column. Table 1 repeats the results of the table without the approximations, and also identifies where each equation appears in [2]. The equations in Table 1 still use the Pearson's form of the incomplete gamma function. However, the notation has been changed to denote Pearson's form as $I_P(\cdot,\cdot)$, as in Eqn. (1); the notation $I(\cdot,\cdot)$ will now be reserved for the more common and MATLAB™-compatible form of Eqn. (2).

Table 1. Exact Probability of Detection for Swerling Model Fluctuating Targets with a Square-Law Detector and Pearson's Form of the Incomplete Gamma Function.

Case	P_D	Comments
0 or 5	$Q_M\left(\sqrt{2N\bar{\chi}},\sqrt{2T}\right)+e^{-(T+N\bar{\chi})}\sum_{r=2}^{N}\left(\frac{T}{N\bar{\chi}}\right)^{\frac{r-1}{2}} I_{r-1}\left(2\sqrt{NT\bar{\chi}}\right)$	Nonfluctuating case. $Q_M(\cdot,\cdot)$ is Marcum's Q function. M&M eqn. (3-37).
1	$1-I_P\left[\frac{T}{\sqrt{N-1}},N-2\right]+\ldots$ $\left(1+\frac{1}{N\bar{\chi}}\right)^{N-1}\exp\left[\frac{-T}{1+N\bar{\chi}}\right]I_P\left[\frac{T}{\left(1+(1/N\bar{\chi})\right)\sqrt{N-1}},N-2\right]$	M&M eqn. (3-56).
2	$1-I_P\left[\frac{T}{(1+\bar{\chi})\sqrt{N}},N-1\right]$	M&M eqn. (3-60).
3	$N=1 \text{ or } 2:\ \left(1+\frac{1}{(N\bar{\chi}/2)}\right)^{N-2}\bullet\ldots$ $\left[1+\frac{T}{1+(N\bar{\chi}/2)}-\frac{N-2}{N\bar{\chi}/2}\right]\exp\left[\frac{-T}{1+(N\bar{\chi}/2)}\right]$ $N>2:\ \frac{T^{N-1}e^{-T}c}{(N-2)!}+\sum_{l=0}^{N-2}\frac{e^{-T}T^l}{l!}+\frac{e^{-cT}}{(1-c)^{N-2}}\bullet\ldots$ $\left[1-\frac{(N-2)c}{(1-c)}+cT\right]\left[1-\sum_{l=0}^{N-2}\frac{e^{-(1-c)T}T^l(1-c)^l}{l!}\right]$	M&M eqns. (3-69) and (A-85) $c\equiv\frac{1}{1+(N\bar{\chi}/2)}$
4	$c^N\sum_{k=0}^{N}\frac{N!}{k!(N-k)!}\left(\frac{1-c}{c}\right)^{N-k}\left\{\sum_{l=0}^{2N-1-k}\frac{e^{-cT}(cT)^l}{l!}\right\},\quad T>N(2-c)$ $1-c^N\sum_{k=0}^{N}\frac{N!}{k!(N-k)!}\left(\frac{1-c}{c}\right)^{N-k}\sum_{l=2N-k}^{\infty}\frac{e^{-cT}(cT)^l}{l!},\quad T<N(2-c)$	$c\equiv\frac{1}{1+(\bar{\chi}/2)}$ M&M eqns. (A-107) and (A-111)
$P_{FA}=1-I_P\left(\frac{T}{\sqrt{N}},N-1\right)$ in all cases		M&M eqn. (2-17)
$I_P(\cdot,\cdot)$ is Pearson's form of the incomplete gamma function; $I_k(\cdot)$ is the modified Bessel function of the first kind and order k.		

In order to more easily translate these expressions into computer code, Table 2 restates Table 1 in terms of the MATLAB™-compatible normalized incomplete gamma function $I(\cdot,\cdot)$. Note that this affects only the Swerling 1 and 2 expressions, and the P_{FA} expression. In addition to retaining which M&M equation is the source of each case, we also note where the same expressions can be found in Barton [5] for the Swerling 1 and 2 cases.

Table 2. Exact Probability of Detection for Swerling Model Fluctuating Targets with a Square-Law Detector and the MATLAB™-Compatible Form of the Incomplete Gamma Function.

Case	P_D	Comments
0 or 5	$Q_M\left(\sqrt{2N\bar{\chi}},\sqrt{2T}\right)+e^{-(T+N\bar{\chi})}\sum_{r=2}^{N}\left(\frac{T}{N\bar{\chi}}\right)^{\frac{r-1}{2}} I_{r-1}\left(2\sqrt{NT\bar{\chi}}\right)$	Nonfluctuating case. $Q_M(\cdot,\cdot)$ is Marcum's Q function. M&M eqn. (3-37)
1	$1-I(T,N-1)+\ldots$ $\left(1+\frac{1}{N\bar{\chi}}\right)^{N-1}\exp\left[\frac{-T}{1+N\bar{\chi}}\right] I\left[\frac{T}{(1+(1/N\bar{\chi}))},N-1\right]$	Equivalent to M&M eqn. (3-56) Barton eqn. (4.28)
2	$1-I\left[\frac{T}{(1+\bar{\chi})},N\right]$	Equivalent to M&M eqn. (3-60). Barton eqn. (4.35)
3	$N=1\text{ or }2:\ \left(1+\frac{1}{(N\bar{\chi}/2)}\right)^{N-2}\bullet\ldots$ $\left[1+\frac{T}{1+(N\bar{\chi}/2)}-\frac{N-2}{N\bar{\chi}/2}\right]\exp\left[\frac{-T}{1+(N\bar{\chi}/2)}\right]$ $N>2:\ \frac{T^{N-1}e^{-T}c}{(N-2)!}+\sum_{l=0}^{N-2}\frac{e^{-T}T^{l}}{l!}+\frac{e^{-cT}}{(1-c)^{N-2}}\bullet\ldots$ $\left[1-\frac{(N-2)c}{(1-c)}+cT\right]\left[1-\sum_{l=0}^{N-2}\frac{e^{-(1-c)T}T^{l}(1-c)^{l}}{l!}\right]$	M&M eqns. (3-69) and (A-85) $c\equiv\frac{1}{1+(N\bar{\chi}/2)}$
4	$c^{N}\sum_{k=0}^{N}\frac{N!}{k!(N-k)!}\left(\frac{1-c}{c}\right)^{N-k}\left\{\sum_{l=0}^{2N-1-k}\frac{e^{-cT}(cT)^{l}}{l!}\right\}\quad T>N(2-c)$ $1-c^{N}\sum_{k=0}^{N}\frac{N!}{k!(N-k)!}\left(\frac{1-c}{c}\right)^{N-k}\sum_{l=2N-k}^{\infty}\frac{e^{-cT}(cT)^{l}}{l!}\quad T<N(2-c)$	$c\equiv\frac{1}{1+(\bar{\chi}/2)}$ M&M eqns. (A-107) and (A-111)
$P_{FA}=1-I(T,N)$ in all cases		Equivalent to M&M eqn. (2-17) Barton eqn. (4.4)
$I(\cdot,\cdot)$ is the normalized incomplete gamma function; $I_k(\cdot)$ is the modified Bessel function of the first kind and order k.		

3 Simplified Equations for P_D for Some Swerling Cases

Properties of the normalized incomplete gamma function can be used to simplify the expressions for P_D in a few cases, producing what are certainly better-known and more calculator-friendly expressions. These expressions are exact for some Swerling cases and values of N, and approximate for others, as noted in the discussion.

First, note the following properties of $I(\cdot,\cdot)$ [4]:

- For any c, $I(c,0)=1$
- For any c, $I(c,1)=1-e^{-c}$
- For any c, $I(c,2)=1-(1+c)e^{-c}$
- For $d > 0$, $\lim_{c\to\infty} I(c,d)=1$

3.1 Swerling 1 Case

When N = 1, applying the property $I(c,0)=1$ immediately reduces the Swerling 1 expression in Table 2 to the much simpler but still exact expression

$$P_D = \exp\left[\frac{-T}{1+\bar{\chi}}\right] \qquad \text{[Swerling 1, } N = 1\text{; M\&M Eqn. (3-52)]} \tag{4}$$

Also in the N = 1 case, the property $I(c,1)=1-\exp(-c)$ can be applied to the expression for P_{FA} to get

$$P_{FA} = e^{-T} \qquad [N = 1\ ] \tag{5}$$

Eliminating T from eqns. (4) and (5) gives, for a single sample of a Swerling 1 target in noise,

$$P_D = P_{FA}^{\ 1/(1+\bar{\chi})} \qquad \text{[Swerling 1, } N = 1\text{]} \tag{6}$$

Now assume that $P_{FA} \ll 1$ and $N\bar{\chi} > 2$. (The quantity $N\bar{\chi}$ can be thought of (very) crudely as the integrated signal-to-noise ratio at the input to the square law detector.) DiFranco and Rubin argue on p. 390 of [6] that both of these conditions must hold true if we are going to have any chance of detecting a target while maintaining a small P_{FA}. They further argue, apparently relying on the property $\lim_{c\to\infty} I(c,d)=1$, that the two incomplete gamma functions in the Swerling 1 exact result are both approximately equal to one. Under these conditions, the exact expression in the table then reduces to

$$P_D \approx \left(1+\frac{1}{N\bar{\chi}}\right)^{N-1} \exp\left[\frac{-T}{1+N\bar{\chi}}\right] \qquad \text{[Swerling 1; } P_{FA} \ll 1 \text{ and } N\bar{\chi} > 2\text{; Barton eqn. (4.30)]} \tag{7}$$

This result is exact (and matches Eqn. (4)) for N = 1.

3.2 Swerling 3 Case

Using the same arguments as in the Swerling 1 case, DiFranco and Rubin show on p. 421 of [6] that the exact expression given in Table 2 for the Swerling 3 case with N = 1 or 2 is a valid approximation for larger N as well, provided again that $P_{FA} \ll 1$ and $N\bar{\chi} > 2$ and using $\lim_{c\to\infty} I(c,d)=1$. Therefore,

$$P_D \approx \left(1+\frac{1}{\left(N\bar{\chi}/2\right)}\right)^{N-2}\left[1+\frac{T}{1+\left(N\bar{\chi}/2\right)}-\frac{N-2}{N\bar{\chi}/2}\right]\exp\left[\frac{-T}{1+\left(N\bar{\chi}/2\right)}\right] \quad (8)$$

[Swerling 3; $P_{FA} \ll 1$ and $N\bar{\chi} > 2$; M&M Eqn. (3-69), Barton eqn. (4.38)]

Again, this expression is exact for $N = 1$ and $N = 2$.

4 Relationships Between Certain Swerling Cases

4.1 Fluctuation Models Are Moot When $N = 1$

The effect of noncoherent integration is moot when $N = 1$, i.e. there are not multiple samples to integrate. Consequently, the Swerling 1 and 2 cases, which share the same exponential target probability density function (PDF), should produce identical results in the $N = 1$ case. Similarly, the Swerling 3 and 4 cases, which share a chi-square target PDF, should be identical when $N = 1$.

The equivalence of Swerling 1 and 2 is easily shown. It was seen in Eqn. (4) that $P_D = \exp\left[-T/\left(1+\bar{\chi}\right)\right]$ for $N = 1$ in the Swerling 1 case. Using $I(c,1) = 1-e^{-c}$ shows that the Swerling 2 result is identical, as expected. The equivalence of the Swerling 1 and 2 cases when $N = 1$ also means that Eqn. (6) applies to the Swerling 2 case as well.

Concerning the equivalence of Swerling 3 and 4 models when $N = 1$, it is trivial to write down the expression for the Swerling 3 case, but this cannot be readily related to the Swerling 4 expressions in Table 2 with $N = 1$. However, it easy to see that the characteristic functions for both cases, which are given in M&M as Eqns. (3-63) (Swerling 3) and (3-71) (Swerling 4) are identical for $N = 1$, so the resulting PDFs and then detection probabilities must also be equal. The reader is referred to [2] for the detailed expressions.

4.2 Swerling 2 = Swerling 3 When $N = 2$

In a previous technical memorandum [7] it was shown using characteristic functions that the receiver operating characteristic (ROC) curves (i.e., P_D for a given P_{FA} and signal-to-noise ratio $\bar{\chi}$) are identical for the Swerling 2 and 3 cases when $N = 2$. This is also confirmed by the exact equations in Table 2. Applying the property $I(c,2) = 1-(1+c)e^{-c}$ to the Swerling 2 expression with $N = 2$ reduces it to $P_D = \left(1+T/\left(1+\bar{\chi}\right)\right)\exp\left[-T/\left(1+\bar{\chi}\right)\right]$. Setting $N = 2$ in the first expression for the Swerling 3 case produces the identical result.

5 References

[1] M. A. Richards, *Fundamentals of Radar Signal Processing*, second edition. McGraw-Hill, 2014.

[2] D. P. Meyer and H. A. Mayer, *Radar Target Detection: Handbook of Theory and Practice*. Academic Press, 1973.

[3] Section 6.5 in M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*. National Bureau of Standards, Applied Mathematics Series, vol. 55.

[4] Section 8.2 in *NIST Handbook of Mathematical Functions*, Olver et al, eds. Cambridge University Press, 2010.

[5] D. K. Barton, *Radar Equations for Modern Radar*. Artech House, 2013.

[6] J. V. DiFranco and W. L. Rubin, *Radar Detection*. SciTech Publishing, 2004.

[7] M. A. Richards, "Swerling 2 = Swerling 3 When N = 2", technical memorandum, July 2014. Available at http://www.radarsp.com.