Exact and Approximate Detection Probability Formulas in *Fundamentals of Radar Signal Processing*

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1 Introduction

Table 6.1 in the text *Fundamentals of Radar Signal Processing*, 2nd ed. [1], is reproduced below. It gives formulas for the probability of false alarm \( P_{FA} \) and the probability of detection \( P_D \) for the four standard Swerling target fluctuation models and the nonfluctuating model (denoted Swerling 0 or 5, as is often done) in additive Gaussian noise when a square law detector and \( N \)-fold noncoherent integration are used. \( \bar{X} \) is the signal-to-noise ratio. The same table also appears as Table 6.1 of the first edition of the book.

Table 6.1 from [1], “Probability of Detection for Swerling Model Fluctuating Targets with a Square-Law Detector”.

<table>
<thead>
<tr>
<th>Case</th>
<th>( P_D )</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 or 5</td>
<td>( Q_M \left( \sqrt{2N \bar{X}}, \sqrt{2T} \right) + e^{-\left(T+N\bar{X}\right)} \sum_{r=2}^{N} \left( \frac{T}{N\bar{X}} \right)^{r-1} I_{r-1}\left(2\sqrt{NT \bar{X}}\right) )</td>
<td>Nonfluctuating case</td>
</tr>
<tr>
<td>1</td>
<td>( \left(1+\frac{T}{1+\bar{X}}\right)^{N-1} \exp \left[\frac{-T}{1+N\bar{X}}\right] )</td>
<td>Approximate for ( P_{FA} \ll 1 ) and ( N\bar{X} &gt; 1 ); exact for ( N = 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 1-I\left[\frac{T}{1+\bar{X}}, N-1\right] )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( \left(1+\frac{2T}{1+\bar{X}}\right)^{N-2} \left[1+\frac{T}{1+(N\bar{X}/2)}\right] \exp \left[\frac{-T}{1+N\bar{X}/2}\right] )</td>
<td>Approximate for ( P_{FA} \ll 1 ) and ( N\bar{X}/2 &gt; 1 ); exact for ( N = 1 ) or 2</td>
</tr>
<tr>
<td>4</td>
<td>( e^{N} \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \frac{\left(1-c\right)^{N-k}}{c} \sum_{l=0}^{2N-1-k} \frac{e^{-c} T^{l}}{l!} ) for ( T &gt; N(2-c) ) ( 1-e^{N} \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \frac{\left(1-c\right)^{N-k}}{c} \sum_{l=2N-k}^{\infty} \frac{e^{-c} T^{l}}{l!} ) for ( T &lt; N(2-c) )</td>
<td>( c = \frac{1}{1+(\bar{X}/2)} )</td>
</tr>
</tbody>
</table>

\( P_{FA} = 1-I\left[\frac{T}{\sqrt{N}}, N-1\right] \) in all cases

\( I(\cdot, \cdot) \) is Pearson’s form of the incomplete Gamma function; \( I_0(\cdot) \) is the modified Bessel function of the first kind and order \( k \).
The particular forms of the detection equations in this table were obtained from the text by Meyer and Mayer (“M&M”) [2]. However, the versions presented in this table, while useful, are a somewhat inconsistent mix of exact and approximate results. The first goal of this memo is to present a more consistent set of exact results, and to show briefly how some approximate, simplified results are then derived.

A second issue is that [2] and therefore the original table uses Pearson’s form of the incomplete gamma function to express some of the results. Pearson’s incomplete gamma function is defined in [2] as

\[ I_p(a,b) = \frac{a^{b+1}}{b!} \int_0^t t^b e^{-t} dt = \frac{1}{\Gamma(b+1)} \int_0^t t^b e^{-t} dt \]  

Meyer and Mayer use only the first version because the argument \( b \) is always integer; the second version is a generalization for non-integer \( b \). Despite the \( I_p(\cdot,\cdot) \) notation adopted here, the table above uses just \( I(\cdot,\cdot) \) for Pearson’s form because that is the way it was presented in [1].

It now is more common to use an incomplete gamma function or “normalized” incomplete gamma function defined as [3][4]

\[ I(c,d) = \frac{1}{\Gamma(d)} \int_0^c t^{d-1} e^{-t} dt = \frac{1}{(d-1)!} \int_0^c t^{d-1} e^{-t} dt \]  

where the second version is for an integer argument \( d \), which will always be our case. Eqn. (2) is also the definition used by MATLAB™ for their \texttt{gammainc} function. The two versions are related by

\[ I_p(a,b) = I(a\sqrt{b+1},b+1) \]

This conversion is also noted in [5]. The second goal of this memo is to restate the tabular results in terms of the more modern, common, and MATLAB™-compatible normalized incomplete gamma function.

2 Exact Equations for \( P_D \), in Original Pearson’s and “Modern” MATLAB™-Compatible Forms

The formulas for \( P_D \) in the Swerling 1 and 3 cases of the original Table 6.1 above are approximations, as noted in the last column. Table 1 repeats the results of the table without the approximations, and also identifies where each equation appears in [2]. The equations in Table 1 still use the Pearson’s form of the incomplete gamma function. However, the notation has been changed to denote Pearson’s form as \( I_p(\cdot,\cdot) \), as in Eqn. (1); the notation \( I(\cdot,\cdot) \) will now be reserved for the more common and MATLAB™-compatible form of Eqn. (2).
Table 1. Exact Probability of Detection for Swerling Model Fluctuating Targets with a Square-Law Detector and Pearson’s Form of the Incomplete Gamma Function.

<table>
<thead>
<tr>
<th>Case</th>
<th>( P_D )</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 or 5</td>
<td>( Q_M \left( \sqrt{2N\bar{X}}, \sqrt{2T} \right) + e^{-(T+N\bar{X})} \sum_{r=2}^{N} \left( \frac{T}{N\bar{X}} \right)^{r-1} I_{r-1} \left( 2\sqrt{N\bar{X}} \right) )</td>
<td>Nonfluctuating case. ( Q_M \left( \cdot, \cdot \right) ) is Marcum’s ( Q ) function. M&amp;M eqn. (3-37).</td>
</tr>
<tr>
<td>1</td>
<td>( 1 - IP \left[ \frac{T}{\sqrt{N-1}}, N-2 \right] ) + ...</td>
<td>M&amp;M eqn. (3-56).</td>
</tr>
<tr>
<td>2</td>
<td>( 1 - IP \left[ \frac{T}{(1+\bar{X})\sqrt{N}}, N-1 \right] )</td>
<td>M&amp;M eqn. (3-60).</td>
</tr>
<tr>
<td>3</td>
<td>( N = 1 \text{ or } 2: \left[ 1 + \frac{1}{(N\bar{X}/2)} \right]^{N-2} \cdot \ldots ) ( \sum_{l=0}^{N-2} \frac{e^{-T}T^l}{l!} + e^{-cT} \left( 1-c \right)^{N-2} \cdot \ldots )</td>
<td>M&amp;M eqns. (3-69) and (A-85)</td>
</tr>
<tr>
<td>4</td>
<td>( c^N \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \left( \frac{1-c}{c} \right)^{N-k} \left( \frac{2N-1-k}{c} \right)^{N-k} \cdot \ldots ) ( 1 - c^N \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \left( \frac{1-c}{c} \right)^{N-k} \left( \frac{e^{-cT}T^l}{l!} \right)^{N-k} \cdot \ldots )</td>
<td>M&amp;M eqns. (A-107) and (A-111)</td>
</tr>
</tbody>
</table>

\( IP \left( \cdot, \cdot \right) \) is Pearson’s form of the incomplete gamma function; \( I_k \left( \cdot, \cdot \right) \) is the modified Bessel function of the first kind and order \( k \).

In order to more easily translate these expressions into computer code, Table 2 restates Table 1 in terms of the MATLAB\textsuperscript{\textregistered}-compatible normalized incomplete gamma function \( I \left( \cdot, \cdot \right) \). Note that this affects only the Swerling 1 and 2 expressions, and the \( P_{FA} \) expression. In addition to retaining which M&M equation is the source of each case, we also note where the same expressions can be found in Barton [5] for the Swerling 1 and 2 cases.
Table 2. Exact Probability of Detection for Swerling Model Fluctuating Targets with a Square-Law Detector and the MATLAB™-Compatible Form of the Incomplete Gamma Function.

<table>
<thead>
<tr>
<th>Case</th>
<th>PD</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 or 5</td>
<td>( Q_M \left( \sqrt{2N\gamma}, \sqrt{2T} \right) + e^{-\left(T+N\gamma\right)} \sum_{r=2}^{N} \left( \frac{T}{N\gamma} \right)^{r-1} I_{r-1} \left( 2\sqrt{NT\gamma} \right) )</td>
<td>Nonfluctuating case. ( Q_M \left( \cdot, \cdot \right) ) is Marcum’s ( Q ) function. M&amp;M eqn. (3-37)</td>
</tr>
<tr>
<td>1</td>
<td>( 1 - I \left( T, N-1 \right) + \ldots ) ( \left( 1 + \frac{1}{N\gamma} \right)^{-N} \exp \left[ -\frac{T}{1+N\gamma} \right] I \left( T \left( \frac{T}{1+(1/N\gamma)} \right), N-1 \right) )</td>
<td>Equivalent to M&amp;M eqn. (3-56). Barton eqn. (4.28)</td>
</tr>
<tr>
<td>2</td>
<td>( 1 - I \left[ \frac{T}{(1+\gamma)}, N \right] )</td>
<td>Equivalent to M&amp;M eqn. (3-60). Barton eqn. (4.35)</td>
</tr>
<tr>
<td>( N = 1 ) or ( 2 ): ( \left( 1 + \frac{1}{(N\gamma/2)} \right)^{-N-2} \ldots ) ( \left( 1 + \frac{T}{1+(N\gamma/2)} \right)^{-N-2} \exp \left[ -\frac{T}{1+(N\gamma/2)} \right] )</td>
<td>M&amp;M eqns. (3-69) and (A-85)</td>
<td></td>
</tr>
<tr>
<td>( N &gt; 2 ): ( \frac{T^{N-1} e^{-T}}{(N-2)!} + \sum_{l=0}^{N-2} \frac{e^{-T} T^{l} L_{l}}{l!} + e^{-cT} \ldots ) ( 1 - \frac{(N-2)c}{(1-c)} + cT \left[ 1 - \sum_{l=0}^{N-2} \frac{e^{-(1-c)T} T^{l} (1-c)^{l}}{l!} \right] )</td>
<td>( c \equiv \frac{1}{1+(N\gamma/2)} )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( c^{N} \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \left( \frac{1-c}{c} \right)^{N-k} \sum_{l=0}^{2N-1-k} \frac{e^{-cT} (cT)^{l}}{l!} ) ( T &gt; N(2-c) ) ( c \equiv \frac{1}{1+(\gamma/2)} )</td>
<td>M&amp;M eqns. (A-107) and (A-111)</td>
</tr>
<tr>
<td>( 1 - c^{N} \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \left( \frac{1-c}{c} \right)^{N-k} \sum_{l=2N-k}^{\infty} \frac{e^{-cT} (cT)^{l}}{l!} ) ( T &lt; N(2-c) )</td>
<td>Equivalent to M&amp;M eqn. (2-17). Barton eqn. (4.4)</td>
<td></td>
</tr>
</tbody>
</table>

\( I(\cdot,\cdot) \) is the normalized incomplete gamma function; \( I_k(\cdot) \) is the modified Bessel function of the first kind and order \( k \).

## 3 Simplified Equations for PD for Some Swerling Cases

Properties of the normalized incomplete gamma function can be used to simplify the expressions for \( P_D \) in a few cases, producing what are certainly better-known and more calculator-friendly expressions. These expressions are exact for some Swerling cases and values of \( N \), and approximate for others, as noted in the discussion.

First, note the following properties of \( I(\cdot,\cdot) \) [4]:

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*M. A. Richards, “Exact and Approximate Detection Probability Formulas”* Sep. 26, 2018

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For any \( c \), \( I(c,0) = 1 \)

- For any \( c \), \( I(c,1) = 1 - e^{-c} \)
- For any \( c \), \( I(c,2) = 1 - (1 + c) e^{-c} \)
- For \( d > 0 \), \( \lim_{c \to \infty} I(c,d) = 1 \)

### 3.1 Swerling 1 Case

When \( N = 1 \), applying the property \( I(c,0) = 1 \) immediately reduces the Swerling 1 expression in Table 2 to the much simpler but still exact expression

\[
P_D = \exp \left[ -\frac{T}{1 + \chi} \right] \quad \text{[Swerling 1, } N = 1; \text{ M&M Eqn. (3-52) ] (4)}
\]

Also in the \( N = 1 \) case, the property \( I(c,1) = 1 - \exp(-c) \) can be applied to the expression for \( P_{FA} \) to get

\[
P_{FA} = e^{-T} \quad [N = 1]
\] (5)

Eliminating \( T \) from eqns. (4) and (5) gives, for a single sample of a Swerling 1 target in noise,

\[
P_D = P_{FA}^{1/(1 + \chi)} \quad \text{[Swerling 1, } N = 1]\]

(6)

Now assume that \( P_{FA} \ll 1 \) and \( N \chi > 2 \). (The quantity \( N \chi \) can be thought of (very) crudely as the integrated signal-to-noise ratio at the input to the square law detector.) DiFranco and Rubin argue on p. 390 of [6] that both of these conditions must hold true if we are going to have any chance of detecting a target while maintaining a small \( P_{FA} \). They further argue, apparently relying on the property \( \lim_{c \to \infty} I(c,d) = 1 \), that the two incomplete gamma functions in the Swerling 1 exact result are both approximately equal to one. Under these conditions, the exact expression in the table then reduces to

\[
P_D \approx \left(1 + \frac{1}{N \chi}\right)^{N-1} \exp \left[ -\frac{T}{1 + N \chi} \right] \quad \text{[Swerling 1; } P_{FA} \ll 1 \text{ and } N \chi > 2 \text{; Barton eqn. (4.30) ] (7)}
\]

This result is exact (and matches Eqn. (4)) for \( N = 1 \).

### 3.2 Swerling 3 Case

Using the same arguments as in the Swerling 1 case, DiFranco and Rubin show on p. 421 of [6] that the exact expression given in Table 2 for the Swerling 3 case with \( N = 1 \) or 2 is a valid approximation for larger \( N \) as well, provided again that \( P_{FA} \ll 1 \) and \( N \chi > 2 \) and using \( \lim_{c \to \infty} I(c,d) = 1 \). Therefore,
Again, this expression is exact for \( N = 1 \) and \( N = 2 \).

4 Relationships Between Certain Swerling Cases

4.1 Fluctuation Models Are Moot When \( N = 1 \)

The effect of noncoherent integration is moot when \( N = 1 \), i.e. there are not multiple samples to integrate. Consequently, the Swerling 1 and 2 cases, which share the same exponential target probability density function (PDF), should produce identical results in the \( N = 1 \) case. Similarly, the Swerling 3 and 4 cases, which share a chi-square target PDF, should be identical when \( N = 1 \).

The equivalence of Swerling 1 and 2 is easily shown. It was seen in Eqn. (4) that

\[
P_D \approx \left(1 + \frac{1}{(N \bar{\chi} / 2)}\right)^{N-2} \left[1 + \frac{T}{1 + (N \bar{\chi} / 2)} - \frac{N - 2}{N \bar{\chi} / 2} \right] \exp \left[\frac{-T}{1 + (N \bar{\chi} / 2)}\right]
\]

\[\text{(8)}\]

(Swerling 3; \( P_{FA} \ll 1 \) and \( N \bar{\chi} > 2 \); M&M Eqn. (3-69), Barton eqn. (4.38))

Concerning the equivalence of Swerling 3 and 4 models when \( N = 1 \), it is trivial to write down the expression for the Swerling 3 case, but this cannot be readily related to the Swerling 4 expressions in Table 2 with \( N = 1 \). However, it easy to see that the characteristic functions for both cases, which are given in M&M as Eqns. (3-63) (Swerling 3) and (3-71) (Swerling 4) are identical for \( N = 1 \), so the resulting PDFs and then detection probabilities must also be equal. The reader is referred to [2] for the detailed expressions.

4.2 Swerling 2 = Swerling 3 When \( N = 2 \)

In a previous technical memorandum [7] it was shown using characteristic functions that the receiver operating characteristic (ROC) curves (i.e., \( P_D \) for a given \( P_{FA} \) and signal-to-noise ratio \( \bar{\chi} \)) are identical for the Swerling 2 and 3 cases when \( N = 2 \). This is also confirmed by the exact equations in Table 2. Applying the property \( I(c,1) = 1 - e^{-c} \) to the Swerling 2 expression with \( N = 2 \) reduces it to

\[
P_D = \left(1 + T/(1 + \bar{\chi})\right) \exp[-T/(1 + \bar{\chi})]
\]

Setting \( N = 2 \) in the first expression for the Swerling 3 case produces the identical result.

5 References


