Doppler Shift in Radar

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1 Preface

This technical memorandum is essentially identical to Section 2.6 of the second edition of *Fundamentals of Radar Signal Processing* [1]. A third edition of that text is in preparation at the time of this writing and is expected to be published in 2021.The detailed derivations of the Doppler shift in Section 2.6.1 of the second edition are being removed in the third because the level of detail is greater than appropriate. Reproducing those details in this memo and posting it on the textbook support web site ensures that the more detailed derivation remains available to interested readers. Sections 2.6.2 and 2.6.3 on the stop-and-hop approximation and spatial Doppler remain in the third edition text but are also included here for completeness.

2 Doppler Shift

If a radar and scatterer are not at rest with respect to each other, the frequency F_r of the received echo will differ from the transmitted frequency F_t due to the Doppler effect. Doppler shifts can be used to advantage to detect echoes from moving targets in the presence of much stronger echoes from clutter or to drastically improve cross-range resolution. Uncompensated Doppler shifts can also have harmful effects, particularly a loss of sensitivity for some types of waveforms. Thus, characterization and measurement of Doppler shifts is an important topic in radar.

Consider an arbitrary waveform $\bar{x}(t)$, pulsed or continuous wave (CW), transmitted by a monostatic radar. The waveform is reflected from a perfectly conducting target at an arbitrarily time-varying range $R(t)$. For instance, a constant-range target would have $R(t)$ equal to a fixed R_0 meters, while a constant-velocity target would have $R(t) = R_0 - vt$ meters.[1] It makes no difference whether the radar, the target, or both are moving such that the range between the two is $R(t)$, so it can be assumed without loss of generality that the radar is stationary and the target is moving, and that all measurements are made in the frame of reference of the radar. Under these conditions the received signal can be shown to be [2][3]

$$\bar{y}(t) = -k \cdot \left[1 - 2\dot{h}(t)\right] \cdot \bar{x}\left[2h(t) - t\right] \tag{1}$$

where k absorbs all radar range equation amplitude factors and $h(t)$ is the function that satisfies

[1] It is probably more common to define a constant-velocity target so that positive v corresponds to increasing range, but the preference here is to define v so that a positive v gives a positive Doppler shift.

$$h(t)+\frac{1}{c}R\left[h(t)\right]=t \tag{2}$$

The dot over $h(t)$ in Eq. (1) denotes the time derivative. The minus sign (180° phase shift) is required by the boundary conditions at a perfectly conducting surface. The function $h(t)$, which has units of seconds, is the time at which a wave must have been launched in order to intercept the moving target at time t and range $R(t)$. For example, if $R(t)$ is a constant R_0, then $h(t) = t - R_0/c$.

For instantaneous velocities $\dot{R}(t)$ that are a small fraction of the speed of light (virtually always the case as will be discussed shortly), the "quasi-stationary" assumption is commonly made. This holds that the range change during the short flight of any particular point in the waveform from the transmitter to the target is negligible. Then $R\left[h(t)\right]\approx R(t)$ so that [2]

$$\begin{aligned} h(t) &\approx t-\frac{1}{c}R(t) \\ \bar{y}(t) &\approx -k\cdot\left[1-2\dot{h}(t)\right]\bar{x}\left[t-\frac{2R(t)}{c}\right]=-k\cdot\left[\frac{2\dot{R}(t)}{c}-1\right]\bar{x}\left[t-\frac{2R(t)}{c}\right] \\ &\approx k\cdot\bar{x}\left[t-\frac{2R(t)}{c}\right] \end{aligned} \tag{3}$$

The last step also uses the assumption $\dot{R}(t) << c$. This result is exact when the target is stationary, $R(t) = R_0$. Then $h(t) = t - R_0/c$ exactly and $\bar{y}(t)=k\cdot\bar{x}(t-2R_0/c)$ exactly.

The case of a constant-velocity target is of special interest. Returning to the exact result of Eqs. (1) and (2), let $R(t) = R_0 - vt$ and define $\beta_v \equiv v/c$. It is easy to show that [2]

$$\begin{aligned} h(t) &= \frac{1}{1-\beta_v}\left(t-\frac{R_0}{c}\right) \\ \left[1-2\dot{h}(t)\right] &= -\frac{1+\beta_v}{1-\beta_v}\equiv-\alpha_v \end{aligned} \tag{4}$$

so that [4][5][6]

$$\bar{y}(t)=k\cdot\alpha_v\cdot\bar{x}\left[\alpha_v\left(t-\frac{2R_0}{(1+\beta_v)c}\right)\right] \tag{5}$$

Assume the transmitted waveform is a standard narrowband RF pulse or CW signal. It can be written

$$\bar{x}(t)=A(t)\exp\left[j\left(2\pi F_t t+\phi_0\right)\right] \tag{6}$$

where $A(t)$ is the amplitude modulation function, typically a constant A for a CW waveform and a square pulse of amplitude A and duration τ for a pulsed signal. The received echo waveform will be

$$\begin{aligned}\bar{y}(t) &= k\cdot\alpha_v\cdot A\left(\alpha_v\left[t-\frac{2R_0}{(1+\beta_v)c}\right]\right)\exp\left(j\left[2\pi F_t\alpha_v\left(t-\frac{2R_0}{(1+\beta_v)c}\right)+\phi_0\right]\right)\\ &= k\cdot\alpha_v\cdot A\left[\alpha_v t-\frac{2R_0}{(1-\beta_v)c}\right]\exp\left(j\left[2\pi\alpha_v F_t t-\frac{4\pi R_0}{(1-\beta_v)\lambda}+\phi_0\right]\right)\end{aligned} \tag{7}$$

Inspection of Eq. (7) reveals several characteristics of the received signal. Its frequency is $\alpha_v F_t$ Hz. The *change* in frequency is the Doppler shift F_D:

$$F_D = \alpha_v F_t - F_t = (\alpha_v - 1)F_t = \frac{2v}{(1-\beta_v)\lambda}\ \text{Hz} \tag{8}$$

The Doppler shift is positive for approaching targets $\left(v>0 \Rightarrow \beta_v > 0 \Rightarrow \alpha_v > 0\right)$ and negative for receding targets as expected. The phase of the received signal is decreased by

$$\Delta\phi = -\frac{4\pi R_0}{(1-\beta_v)\lambda}\ \text{radians} \tag{9}$$

The waveform is scaled in time by the factor α_v. For an approaching target $\alpha_v > 1$ so that a transmitted signal is compressed by a factor of α_v on reception; for a receding target it is lengthened by a factor of α_v. The compression (expansion) of the signal in time results in an expansion (compression) of the signal bandwidth by the factor α_v due to the "scaling" or "reciprocal spreading" property of Fourier transforms [7]. Finally, the amplitude of the waveform is scaled by the factor α_v (in addition to the range equation effects), a requirement of conservation of energy when the time scale is altered.

It is virtually always the case in radar that the ratio $|\beta_v| = |v/c|$ is very small. For example, a car traveling at 60 mph (26.82 m/s) has a ratio $|v/c|$ of 8.94×10^{-8}; an aircraft at Mach 1 (about 340.3 m/s at sea level) has $|v/c| = 1.13 \times 10^{-6}$; and even a low-earth orbit (LEO) satellite with a velocity of 7800 m/s has $|v/c| = 2.6 \times 10^{-5}$. Expand each of the terms $1/(1\pm\beta_v)$ and $\alpha_v = (1+\beta_v)/(1-\beta_v)$ in a binomial series and retain terms only to first order in β_v:

$$\begin{aligned}\frac{1}{1\pm\beta_v} &= 1\mp\beta_v+\beta_v^2\mp\beta_v^3+\ldots\approx 1\mp\beta_v\\ \alpha_v = \frac{1+\beta_v}{1-\beta_v} &= (1+\beta_v)\left(\frac{1}{1-\beta_v}\right) = (1+\beta_v)\left(1+\beta_v+\beta_v^2+\beta_v^3+\ldots\right)\approx 1+2\beta_v\end{aligned} \tag{10}$$

Equation (5) and the sinusoidal special case of Eq. (7) then reduce to

$$\begin{aligned}\bar{y}(t) &\approx k\cdot(1+2\beta_v)\bar{x}\left[(1+2\beta_v)t-(1+\beta_v)\frac{2R_0}{c}\right]\\ &= k\cdot(1+2\beta_v)A\left[(1+2\beta_v)t-\frac{2(1+\beta_v)R_0}{c}\right]\exp\left(j\left[2\pi(1+2\beta_v)F_t t-\frac{(1+\beta_v)4\pi R_0}{\lambda}+\phi_0\right]\right)\\ &\qquad \text{(sinusoidal signal)}\end{aligned} \tag{11}$$

If the signal is a pulse of length τ, the echoed pulse length will be $\tau' = \tau/\alpha_v \approx (1-+2\beta_v)\tau$. This small change in the pulse duration of $2\beta_v\tau$ seconds is insignificant and can be ignored. The amplitude factor of $(1+2\beta_v)$ is certainly negligible compared to range equation effects and can also be ignored. The change in delay from $2R_0/c$ to $2(1+2\beta_v)R_0/c$ is also usually insignificant, though for a system with fine range resolution at long enough ranges the error could become a significant fraction of a range resolution cell. However, β_v cannot be neglected in the phase term because the factor of $4\pi\beta_v R_0/\lambda$ will frequently be a large fraction or even a multiple of π. With these three approximations to the envelope term and amplitude, the Doppler shift effects on the sinusoidal pulse of Eq. (11) reduce to

$$\begin{aligned}\bar{y}(t) &\approx k\cdot A\left(t-\frac{2R_0}{c}\right)\exp\left(j\left[2\pi(F_t+2\beta_v F_t)t-(1+\beta_v)\frac{4\pi R_0}{\lambda}+\phi_0\right]\right)\\ &= k\cdot A\left(t-\frac{2R_0}{c}\right)\exp\left(-j(1+\beta_v)\frac{4\pi}{\lambda}R_0\right)\exp\left[+j2\pi\left(\frac{2v}{\lambda}\right)t\right]\exp\left[j\left(2\pi F_t t+\phi_0\right)\right]\end{aligned} \tag{12}$$

The key result is that, to an excellent approximation, the pulse echoed from a constant-velocity target exhibits a Doppler shift of $2vF_t/c = 2v/\lambda$ Hz and a phase shift of $-(1+\beta_v)(4\pi/\lambda)R_0$ radians.

The numerical values of Doppler shift are small compared to the radar frequencies. Table 1 gives the magnitude of the Doppler shift corresponding to a velocity of 1 m/s at various radar frequencies. The Mach 1 aircraft observed with the L band radar would cause a Doppler shift of only 2.27 kHz in the 1 GHz carrier frequency.

For a monostatic radar and a constant-velocity target, the observed Doppler shift is proportional to the component of velocity in the direction of the radar, called the *radial velocity*. Consider the geometry of the two examples illustrated in two dimensions in Fig. 1. The aircraft is traveling at v m/s in the direction shown. At the instant shown, the angle between its velocity vector and the *line of sight* (LOS) vector from the radar position to the target position (sometimes called the *cone angle*) is ϕ. The radial velocity component along the LOS is $v\cos\phi$ meters per second. The magnitude of the Doppler shift is maximum when the target is traveling directly toward or away from the radar ($\phi = \pi$ or $-\pi$ radians). The Doppler shift is zero regardless of the target velocity when the target is crossing orthogonally to the radar boresight ($\phi = \pi/2$ radians).

Table 1. Doppler Shift Resulting from a Velocity of 1 m/s

Band	Frequency, GHz	Doppler Shift (Hz) for v = 1 m/s
L	1	6.67
C	6	40.0
X	10	66.7
K_a	35	233
W	95	633

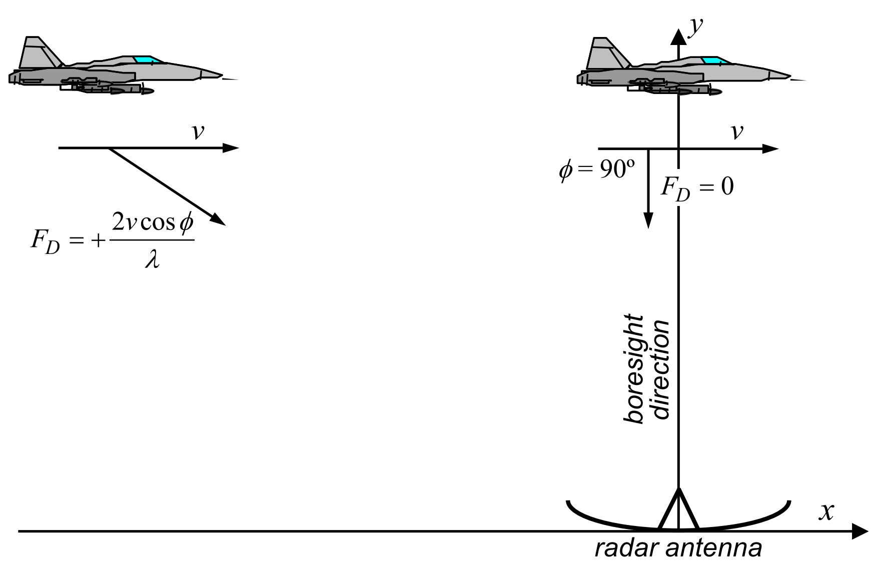

Figure 1. Doppler shift is determined by the radial component of relative velocity between the target and radar.

Equations (1) and (2) can be solved for the exact behavior of other regular patterns of radar-target motion as well. The solution for constant acceleration is given in [3]. Even where a closed form solution for $h(t)$ is difficult or impossible to find, it can still be developed using an iterative approximation approach.

3 The Stop-and-Hop Approximation and Phase History

The quasi-stationary assumption of Eq. (3) provides a simplified but very useful model of reflection of a radar signal from a target moving relative to the radar. Apply it to the pulsed waveform $A(t)\exp[j(2\pi F_t t+\phi_0)]$ where $A(t)$ is a τ-second square pulse. Using the same envelope approximations employed to obtain Eq. (12) gives

$$\begin{aligned}\bar{y}(t) &= k\cdot A\left(t-\frac{2R(t)}{c}\right)\exp\left(j\left[2\pi F_t\left(t-\frac{2R(t)}{c}\right)+\phi_0\right]\right)\\ &\approx k\cdot A\left(t-\frac{2R_0}{c}\right)\exp\left(-j\frac{4\pi}{\lambda}R(t)\right)\exp\left[j\left(2\pi F_t t+\phi_0\right)\right]\end{aligned} \tag{13}$$

where R_0 is the initial range at the time of pulse transmission. Equation (13) states that the echo is received with a time delay corresponding to the range at the beginning of the pulse transmission but with a phase modulation related to the time variation in range. This is the "stop" part of the *stop-and-hop* assumption common in radar analysis: the envelope of the echo appears as if the target motion effectively stopped while the pulse was in transit. The "hop" portion will be discussed shortly.

Equation (13) adequately describes not only constant but also time-varying Doppler frequency shifts. If the target moves relative to the radar at constant velocity, $R(t) = R_0 - vt$,

$$\bar{y}(t) = k\cdot A\left(t-\frac{2R_0}{c}\right)\exp\left(-j\frac{4\pi}{\lambda}R_0\right)\exp\left[j2\pi\left(\frac{2v}{\lambda}\right)t\right]\exp\left[j\left(2\pi F_t t+\phi_0\right)\right] \tag{14}$$

Equation (14) is identical to the second line of Eq. (12), with the exception that the constant phase shift is $-4\pi R_0/\lambda$ instead of $-(1+\beta_v)4\pi R_0/\lambda$ radians. This difference in the constant phase shift does not affect the magnitude or Doppler frequency shift of the echo and can be ignored. Thus the analysis approach of Eq. (3) is consistent with the earlier results in all important respects.

For a more interesting example of the use of Eq. (3), consider Fig. 1 again. Let the radar be located at (x,y) coordinates $(x_r = 0,\ y_r = 0)$ with its antenna aimed in the $+y$ direction, and let the coordinates of the target aircraft be $(x_t = vt,\ y_t = R_0)$. This means that the target aircraft is on the radar boresight at a range R_0 at time $t = 0$ and is crossing orthogonal to the radar line of sight at a velocity v meters per second. The range between radar and aircraft is

$$R(t) = \sqrt{R_0^2+(vt)^2} = R_0\sqrt{1+\left(\frac{vt}{R_0}\right)^2} \tag{15}$$

While it is possible to work with Eq. (15) directly, it is common to expand the square root in a power series:

$$R(t)=R_0\left[1+\frac{1}{2}\left(\frac{vt}{R_0}\right)^2-\frac{3}{8}\left(\frac{vt}{R_0}\right)^4-\ldots\right] \tag{16}$$

In evaluating this expression, the range of t that must be considered may be limited by any of several factors, such as the time the target is within the radar main beam or the coherent processing interval duration over which signals will be collected for subsequent processing.

Assume that the distance traveled by the target within this time of interest is much less than the nominal range R_0 so that higher-order terms in (vt/R_0) can be neglected:

$$R(t)\approx R_0+\left(\frac{v^2}{2R_0}\right)t^2 \tag{17}$$

Equation (17) shows that the range is approximately a quadratic function of time for the crossing target scenario of Fig. 1. Using this truncated series in Eq. (13) gives

$$\bar{y}(t)\approx k\cdot A\left(t-\frac{2R_0}{c}\right)\exp\left(-j\frac{4\pi}{\lambda}R_0\right)\exp\left[-j2\pi\left(\frac{v^2}{R_0\lambda}\right)t^2\right]\exp\left[j\left(2\pi F_t t+\phi_0\right)\right] \tag{18}$$

All of the terms are the same as in the constant-velocity case of Eq. (14) except for the middle exponential. Recall that instantaneous frequency is proportional to the time derivative of phase [1]. The quadratic phase function therefore represents a Doppler frequency shift $F_D(t)$ that varies linearly with time due to the changing radar-target geometry:

$$F_D(t)=\frac{1}{2\pi}\frac{d}{dt}\left[-2\pi\left(\frac{v^2}{R_0\lambda}\right)t^2\right]=-\frac{2v^2}{R_0\lambda}t \tag{19}$$

As the target aircraft approaches from the left in Fig. 1 ($t < 0$) the instantaneous Doppler shift is positive. When the aircraft is abreast of the radar ($t = 0$) the Doppler shift is zero because the radial component of velocity is zero. Finally, as the aircraft passes by the radar ($t > 0$) the Doppler shift becomes negative, as would be expected for a receding target. This quadratic range case is important in synthetic aperture radar and is discussed in Chap. 8 of [1].

The exponential term $\exp\left(-j4\pi R(t)/\lambda\right)$ in Eq. (2.98) is called the *phase history* of the received signal. This terminology is applied both to the complex exponential and to just its phase function $-4\pi R(t)/\lambda$. The phase history encodes the variation of the range between the target and radar during the data collection time. For the constant-velocity example [Eq. (14)], the phase history is a linear function of time corresponding to a constant frequency sinusoid, i.e., a constant Doppler shift. For the crossing target example of Eq. (18), it is a quadratic function of time, producing a Doppler shift sinusoid having a frequency that varies linearly with time. Other radar-target motions will produce other functional forms for the phase history.

More generally, the term phase history can refer to the variation of phase or the corresponding complex exponential in any dimension of the radar data. Two other common uses are to describe the fast-time phase function of a frequency- or phase-modulated waveform or the spatial phase variation across the face of an array antenna at a fixed time. As will be seen, phase history is central to radar signal processing. The design of many important radar signal processing operations depends critically on accurately modeling or estimating the phase history of the collected data. Examples include pulse compression, adaptive interference cancellation, and imaging.

4 Measuring Doppler Shift: Spatial Doppler

The Doppler shifts observed in a pulsed radar are too small to be measured from a single pulse echo in most cases. In Chap. 7 of [1] it is shown that a lower bound on the standard deviation of the error in measuring the frequency of a complex sinusoid with unknown amplitude, frequency, and phase using a discrete Fourier transform (DFT) and an observation of length T_{obs} seconds at an integrated SNR in the DFT of χ is $\sigma_F = \sqrt{6/(2\pi)^2 \chi T_{\text{obs}}^2}$ Hz. Applying this to measuring Doppler, this value must be much less than the Doppler shift, $\sigma_F \ll F_D$, if that shift is to be measured with reasonable precision. This leads to a requirement that $T_{\text{obs}} >> \sqrt{6/(2\pi)^2 \chi F_D^2}$. Even for a rather high Doppler shift of 10 kHz and a very good SNR of 30 dB ($\chi = 1000$), T_{obs} must be much larger than 123 μs. To measure the Doppler shift with a single pulse would therefore require pulse lengths greater than 1 ms, much longer than the sub-millisecond (usually less than 100 μs) pulse lengths typically used. For a 1 kHz Doppler shift and 20 dB SNR, a pulse longer than 10 ms would be needed. For this reason, most radars do not measure Doppler shift on an intrapulse basis, although a few designed for very high speed targets (satellites and missiles) and using very long pulses can do so.

The long observation time needed can be obtained by using multiple pulses. Suppose a series of M distinct pulses of duration τ are transmitted beginning at times $t_m = mT$, where T is the pulse repetition interval (PRI). The m^{th} transmitted pulse and received echo (using the quasi-stationary assumption) are

$$\bar{x}_m(t) = A(t - mT)\exp\left[j(2\pi F t + \phi_0)\right] \tag{20}$$

$$\bar{y}_m(t) \approx k' \cdot A\left[t - mT - \frac{2R(mT)}{c}\right]\exp\left\{j\left[2\pi F_t\left(t - \frac{2R(t)}{c}\right) + \phi_0\right]\right\} \tag{21}$$

After demodulation, the baseband received signal is

$$y_m(t) \approx k' \cdot A\left(t - mT - \frac{2R(mT)}{c}\right)\exp\left[-j\frac{4\pi}{\lambda}R(t)\right] \tag{22}$$

where k' includes the $\exp(-j\phi_0)$ term. Assume each baseband pulse echo is sampled $2R_s/c$ seconds after transmission, corresponding to a range R_s. Also assume a target is present within the range bin corresponding to that sample time for the entire data collection time of mT seconds, meaning that $R(t)$ remains in the range interval $[R_s - c\tau/2, R_s]$.[2] The m^{th} sample in this range bin is then

$$\begin{aligned} y_m\left(mT + \frac{2R_s}{c}\right) &= k' \cdot A\left(\frac{2}{c}\left[R_s - R(mT)\right]\right)\exp\left[-j\frac{4\pi}{\lambda}R\left(mT + \frac{2R_s}{c}\right)\right] \\ &= \hat{k} \cdot \exp\left[-j\frac{4\pi}{\lambda}R\left(mT + \frac{2R_s}{c}\right)\right] \\ &\equiv y[m] \end{aligned} \tag{23}$$

The constant $\hat{k}$ combines k' and the amplitude of the sampled pulse envelope $A(\cdot)$. The series of sampled echoes $y[m]$ forms the slow-time series of samples for that range bin, as is described in Chap. 3 of [1].

The "stop" assumption applied in Eq. (13), when used across a series of pulses as in (23), is called the *stop-and-hop* approximation. Relative to the radar, the target is assumed to "stop" at the time of each pulse transmission at the corresponding range $R(mT)$ and then "hop" to the range at the next pulse transmission time, rather than moving continuously.

Consider again a constant velocity target, $R(t) = R_0 - vt$. The slow-time data series becomes

$$\begin{aligned} y[m] &= \hat{k} \cdot \exp\left[-j\frac{4\pi}{\lambda}\left(R_0 - v\left(mT + \frac{2R_s}{c}\right)\right)\right] \\ &= \hat{k} \cdot \exp\left[-j\frac{4\pi}{\lambda}\left(R_0 - 2\beta_v R_s\right)\right]\exp\left[+j2\pi\left(\frac{2v}{\lambda}\right)mT\right] \end{aligned} \tag{24}$$

The first exponential in Eq. (24) is a constant phase shift for all of the slow-time samples $y[m]$ and is of little consequence. The second exponential is a discrete complex sinusoid with normalized frequency $2vT/\lambda$ cycles/sample, corresponding to the expected Doppler frequency of $2v/\lambda$ Hz. Thus, the phase history obtained from a moving target using a series of pulses provides a way to measure the Doppler shift with good precision by observing the signal over an observation time much longer than that of a single pulse.

The manifestation of the target Doppler shift in the slow-time phase history is sometimes referred to as *spatial Doppler*. This terminology emphasizes the fact that the Doppler shift is measured not from intrapulse frequency changes, but rather from the change of phase of the echoes at a given range bin over a series of pulses. Since the echo phase is proportional to range, the succession of pulses effectively measures the change in range over time, which of course is simply velocity and in turn can be scaled into Doppler shift. Because of the inability to measure intrapulse Doppler frequency shifts in most systems,

[2] Movement of a target across multiple range bins during the series of pulses due to high rates of radar-target motion is known as *range migration*. It is much more common in imaging radar due to their much longer observation times, and so is discussed in Chap. 8 of [1]. A means of compensating for range migration is described in [8].

the term "Doppler processing" in radar usually refers to sensing and processing this spatial Doppler information.

5 References

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