Alternative Forms of Albersheim's Equation

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1 Albersheim's Equation

In radar detection theory, the detection probability P_D, false alarm probability P_{FA}, number of samples N noncoherently integrated for a single detection test, and single-sample average signal-to-noise ratio (SNR) χ are closely interrelated. A common problem is to find one of these quantities given the other three. This can be a difficult calculation. For example, in the nonfluctuating case with a linear detector, the target+noise probability density function (PDF) is Rician, a distribution which involves Bessel functions and for which the P_D calculation requires "Marcum's Q function" [1]. While these equations can be solved using modern computational tools such as MATLAB®, an excellent empirical approximation requiring only a scientific calculator known as Albersheim's equation is available for this case. Albersheim's equation relates P_D, P_{FA}, N, and the single-sample SNR in decibels χ_{dB}. As usually presented, Albersheim's "equation" is the series of calculations [2],[3]

$$
\begin{aligned}
A &= \ln\left(\frac{0.62}{P_{FA}}\right), \quad B = \ln\left(\frac{P_D}{1-P_D}\right) \\
\chi_{\text{dB}} &= -5\log_{10} N + \left(6.2 + \frac{4.54}{\sqrt{N+0.44}}\right)\log_{10}\left(A + 0.12AB + 1.7B\right)
\end{aligned}
\tag{1}
$$

The error in the estimated value of χ_{dB} required to obtain specified values of P_D and P_{FA} for a specified N is stated to be less than 0.2 dB for P_{FA} from 10^{-3} to 10^{-7}, P_D from 0.1 to 0.9, and N from 1 to 8096. Albersheim's equation is very useful not only for direct calculations of required SNR but also for such tasks as estimating noncoherent integration gain for the nonfluctuating case [4].[1]

2 Solving Albersheim's Equation for P_D or P_{FA}

As stated, Albersheim's equation determines the average single-sample SNR corresponding to a specified P_D, P_{FA}, and N. However, it can be rearranged to obtain a solution for P_D in terms of the other parameters using the following sequence of calculations:

[1] "Shnidman's equation" plays the same role for the Swerling target models as does Albersheim's equation for the nonfluctuating case, but is somewhat more complicated; see [1].

$$A = \ln\left(\frac{0.62}{P_{FA}}\right), \quad Z = \frac{\chi_{\mathrm{dB}} + 5\log_{10} N}{6.2 + \frac{4.54}{\sqrt{N+0.44}}}, \quad B = \frac{10^Z - A}{1.7 + 0.12A}$$
$$P_D = \frac{1}{1+e^{-B}} \tag{2}$$

A similar rearrangement gives P_{FA} in terms of the other parameters:

$$B = \ln\left(\frac{P_D}{1-P_D}\right), \quad Z = \frac{\chi_{\mathrm{dB}} + 5\log_{10} N}{6.2 + \frac{4.54}{\sqrt{N+0.44}}}, \quad A = \frac{10^Z - 1.7B}{1+0.12B}$$
$$P_{FA} = 0.62e^{-A} \tag{3}$$

Unfortunately, solving the standard form of Albersheim's equation for N in terms of P_D, P_{FA} and χ_{dB} does not appear to be possible because N appears in both logarithm and square root form.

3 Solving Albersheim's Equation for *N*

A method for solving Albersheim's equation for N can be developed by replacing the term $\left(6.2 + 4.54/\sqrt{N+0.44}\right)$ with an approximation that uses $\log_{10}N$, so that N appears in only one functional form and can be isolated. Define $\alpha \equiv \log_{10}N$. Then it can be seen that

$$6.2 + \frac{4.54}{\sqrt{N+0.44}} \approx -\frac{0.4125}{\alpha^2} + \frac{2.4194}{\alpha} + 5.5606 \quad (2 \le N \le 100) \tag{4}$$

This second-order least-squares approximation was found using the `polyfit` function in MATLAB®. Slightly different approximation coefficients result for different ranges of N. For example, using the entire range $2 \le N \le 8096$ over which Albersheim's equation is applied,[2] the coefficients of α^{-1} are (−0.2289, 1.8402, 5.7777). Here we concentrate on the range $2 \le N \le 100$, believing that is a range of more practical interest. Figure 1 illustrates the close fit of the approximation to the Albersheim equation term in Eq. (4).

Using Eq. (4) in Eq. (1) and defining $Z = \log_{10}\left(A + 0.12AB + 1.7B\right)$ gives[3]

$$\chi_{\mathrm{dB}} = -5\alpha + Z\left(-\frac{0.4125}{\alpha^2} + \frac{2.4194}{\alpha} + 5.5606\right) \tag{5}$$

This equation can be rearranged to give a cubic equation in α:

$$5\alpha^3 + \left(\chi_{\mathrm{dB}} - 5.5606 \cdot Z\right)\alpha^2 - 2.4194 \cdot Z\alpha + 0.4125 \cdot Z = 0 \tag{6}$$

[2] Since the expansion is in terms of $1/\log_{10}N$, the N = 1 term must be excluded from the least-squares fit.
[3] This is the same Z as in Eq. (2) or (3).

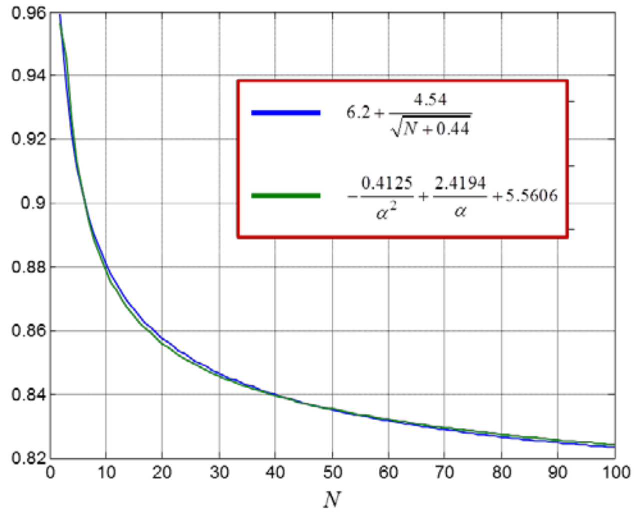


Figure 1. Comparison of Albersheim equation term and its approximation in Eq. (4). $\alpha = \log_{10}N$.

Specifying the values of P_D, P_{FA} and χ_{dB} fixes the value of Z. Equation (6) can then be solved for α. Since Eq. (6) is cubic, it is not trivially solvable with a calculator for the value of α and thus of $\hat{N}$. However, it can be solved numerically (for instance, using the `roots` function in MATLAB®). The largest real root is chosen as the desired value of α. This choice is entirely *ad hoc* but appears to produce good results. N must be a positive integer, therefore the final estimate is

$$\hat{N} = \mathrm{round}\left(\max\left(10^{\alpha}, 1\right)\right) \tag{7}$$

The estimate $\hat{N}$ given in Eq. (7) can be tested by varying values of P_D, P_{FA}, and N over a wide range within the applicability of Albersheim's equation. For each choice of these three parameters, χ_{dB} is computed using Albersheim's equation. Using P_D, P_{FA}, and that computed value of χ_{dB}, Eqs. (6) and (7) are then used to compute $\hat{N}$ and compare it to the true value of N for that case. Figure 2, for example, shows that when P_D = 0.9 and P_{FA} = 10^{-3}, the estimated value of N is correct for $3 \le N \le 84$. The error is −1 for N = 2 (i.e., $\hat{N}$ = 1 in this case) and +1 for $85 \le N \le 100$. In fact, the magnitude of the error does not exceed 1 until N > 560. The maximum percentage error in $\hat{N}$ for this case never exceeds 4.8% for N up to the limit of 8096, even though the approximation coefficients for only the range $2 \le N \le 100$ are used in this sample.

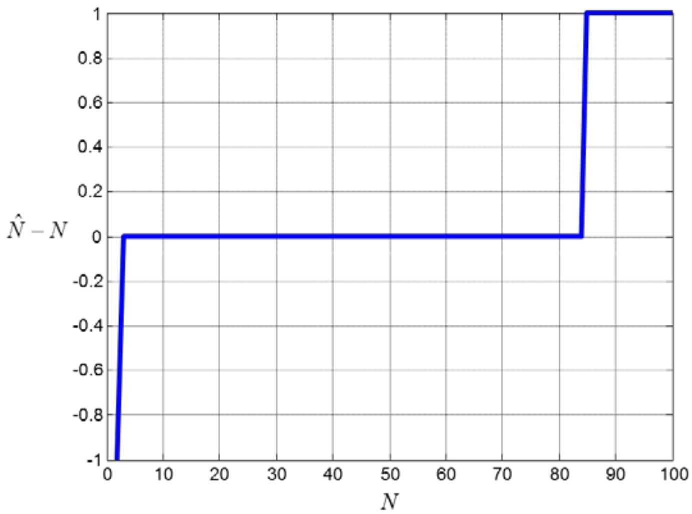
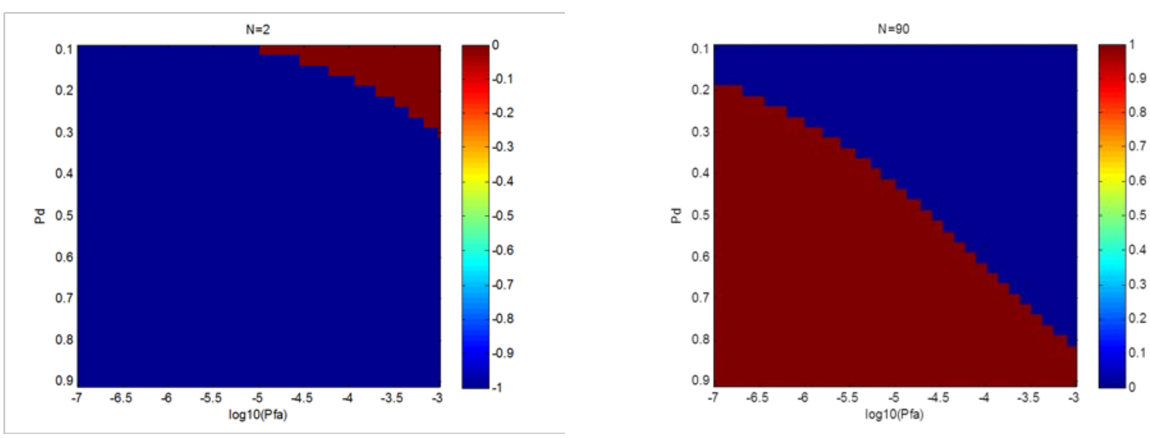


Figure 2. Error in estimate of N when P_D = 0.9 and P_{FA} = 10^{-3}.

For a more complete test, this procedure was carried out as the value of N was varied from 1 to 100; P_D was varied from 0.1 to 0.9 in steps of 0.01; and P_{FA} was varied from 10^{-7} to 10^{-3} in 100 logarithmically-spaced steps. Over this range, the absolute value of the error in $\hat{N}$ never exceeds 1. For N > 2, the percentage error in $\hat{N}$ never exceeds 1.266%, which occurs when P_D = 0.88, P_{FA} = 10^{-7}, and N = 79. (For N =2 the estimated value is often $\hat{N}$ = 1, giving a 50% error.) Figure 3 shows the variation of the absolute error in $\hat{N}$ versus P_D and P_{FA} for N = 2 and N = 90. For all of the similar plots for $3 \le N \le 78$, the error is zero for the full range of P_D and P_{FA} tested.



(a) (b)

Figure 3. Absolute value of error in estimate of N vs. P_D and P_{FA} . (a) N = 2. (b) N = 90. The error is zero for all values of P_D and P_{FA} considered and N between 3 and 78.

4 The Largest Root

Equation (6) has three roots, possibly leading to three different estimates $\hat{N}$ in Eq. (7). Which should be used? In general, a cubic equation with real coefficients such as Eq. (6) has either three real roots, or one real root and a pair of complex conjugate roots. In the discussion so far, the largest real root has been used in Eq. (7) to compute the estimate $\hat{N}$. This choice was *ad hoc* but produces good results.

A numerical search across the same dense grid of P_D, P_{FA}, and N values used to obtain Fig. 3 showed that in practice, both cases occur: three real roots, and one real root plus two complex roots. For values of N of 3 or more, complex roots appear to occur only at the corner case of P_D = 0.1 and $P_{FA} = 10^{-3}$. For N = 2 the complex case occurs more frequently.

Explicit solutions for the roots of a cubic equation are given in [5]. These formulas could be used to compute the roots of Eq. (6) without the use of a numerical solver such as MATLAB®'s `roots`, and then the maximum real root used to compute $\hat{N}$. Such a solution could be implemented, if a bit tediously, on a programmable calculator. If it was possible to predict which of the three roots would be the largest, then the formula for that root would complete a straightforward, deterministic algorithm for estimating N. It is not, unfortunately, obvious how to identify *a priori* which of the three roots will be the largest positive root. This last step remains a topic for additional research. Lacking that result, either a root-finding algorithm such as `roots` or explicit calculation of all three roots using the closed-form equations in [5] must be used.

5 References

[1] M. A. Richards, *Fundamentals of Radar Signal Processing*, second edition. McGraw-Hill, 2014.

[2] Albersheim, W. J., "Closed-Form Approximation to Robertson's Detection Characteristics," *Proceedings IEEE*, vol. 69, no. 7, p. 839, July 1981.

[3] D. W. Tufts and A. J. Cann, "On Albersheim's Detection Equation,: IEEE *Trans. Aerospace & Electronic Systems*, vol. AES-19, no. 4, pp. 644-646, July 1983.

[4] M. A. Richards, "Noncoherent Integration Gain, and its Approximation", technical memorandum, June 9, 2010, revised May 6, 2013. Available at www.radarsp.com.

[5] "Cubic Function", http://en.wikipedia.org/wiki/Cubic_function.