1 Stepped Frequency Data

Suppose an ideal pulsed monochromatic complex sinusoidal signal, $a(t)\exp(j2\pi F t)$ is transmitted and reflects from a point target at range $R_0$. The reflected signal is $a(t-2R_0/c)\exp(j2\pi F(t-2R_0/c))$. This echo signal is demodulated by mixing with the reference signal $\exp(-j2\pi Ft)$ and sampling at time $t = 2R_0/c$, giving the receiver output voltage (a complex scalar) $X = a(0)\exp(-j4\pi F R_0 / c)$. Now suppose we repeat this measurement using a series of $M$ stepped frequencies, $F_m = F_0 + m\Delta F$, $m = 0,1,\ldots,M-1$. Note that the total bandwidth of the data is $B = M\Delta F$ Hz. The series of receiver outputs as a function of the stepped frequency is

$$X[a_j F R c] = \exp \left[ -j4\pi m\Delta F R_0 / c \right] \equiv A \exp \left[ -j4\pi m\Delta F R_0 / c \right]$$

$X[m]$ can be viewed as a set of uniform spectral samples. Consider the $N$-point inverse DFT of $X[m]$ ($N \geq M$; the upper limit on the summation is $M$ instead of $N$ because there are only $M$ spectral samples):

$$x[n] = \frac{1}{N} \sum_{k=0}^{M-1} X[km] e^{j2\pi nm/N}$$

$$= A \frac{1}{N} \sum_{m=0}^{M-1} \exp \left[ j2\pi \left( \frac{n}{N} - \frac{2R_0\Delta F}{c} \right) m \right]$$

(2)

Applying the geometric summation formula gives us a closed form result [2]: 

$$x[n] = \frac{1}{N} \sum_{k=0}^{M-1} X[km] e^{j2\pi nm/N}$$

$$= A \frac{1}{N} \sum_{m=0}^{M-1} \exp \left[ j2\pi \left( \frac{n}{N} - \frac{2R_0\Delta F}{c} \right) m \right]$$

(2)
This familiar “digital sinc” or “asinc” response is the image of the point scatterer at range \( R_0 \).

The peak of the response, \( i.e. \) the sample number \( n_p \) corresponding to \( R_0 \), occurs when the argument of the sine functions equals zero, which occurs when

\[
 n_p = \frac{2R_0\Delta F}{c}
\]  

(4)

Note that to increase \( n_p \) by 1, \( R_0 \) must increase by \( c/2N\Delta F \) meters. Thus, the inverse DFT of the frequency stepped data produces data that is sampled in range at an interval of \( c/2N\Delta F \) meters.

Since the \( N \)-point inverse DFT is periodic with period \( N \) samples (as can be seen from Eq. (2)), the range image generated by the inverse DFT will have an extent of \( N \) times the sample spacing before it repeats; thus the image extent in meters is

\[
 R_{\text{max}} = N\left(\frac{c}{2N\Delta F}\right) = \frac{c}{2\Delta F}
\]  

(5)

There is no guarantee that \( n_p \) will be integer, or will fall within the inverse DFT index range of 0 to \( N-1 \). A non-integer value of \( n_p \) just means that the scatterer range does not fall exactly on one of the range sample points, thus the true peak of the response will be between range samples and a range “straddle loss” [1] will result. The actual value of \( n_p \) can be much larger than \( N-1 \). Consider a scenario where the scene size is 1
km, so that $\Delta F = 150$ kHz (see Eqn. (5)); $N = 256$; and $R_0 = 3500$ m. In this case, $n_p = 896$. Because of the periodicity of the inverse DFT (with period $N$), the actual location of the peak output will be at

$$n_p = \frac{2R_0\Delta FN}{c} \mod N \tag{6}$$

which is $n_p = 128$ in this case. Put another way, the peak index corresponds to the range evaluated modulo $R_{\text{max}}$. In this case, $R_{\text{max}} = 1$ km, so $R_0\mod(R_{\text{max}}) = 3500\mod(1000) = 500$ m. Using either (4) or (6), the resulting peak index is $n_p = 128$.

Next consider the Rayleigh (peak-to-null) resolution of the image of the scatterer. The first null of the response in Eq. (3) occurs when the argument of the sine function in the numerator equals $\pi$. This occurs at sample

$$n_z = N + \frac{2R_0\Delta FN}{c} = n_p + \frac{N}{M} \tag{7}$$

The Rayleigh resolution is the spacing between $n_z$ and $n_p$, and is thus (in samples and, using the sample spacing $c/2N\Delta F$ meters from above, in range)

$$\Delta n = \frac{N}{M} \quad \Rightarrow \quad \Delta R = \frac{c}{2M\Delta F} = \frac{c}{2B} \tag{8}$$

Thus the range resolution follows the usual $c/2B$ rule.

## 2 Range Frequency

Note that the range resolution $\Delta R$ is the inverse of the stepped frequency bandwidth, scaled to spatial frequency units by the factor $2/c$:

$$\Delta R = \frac{1}{(2B/c)} \tag{9}$$

Multiplying the bandwidth by $2/c$ puts it in range frequency units (see [2]). Thus the range resolution in meters is the inverse of the range frequency bandwidth.

## 3 References
